Geometric Algebra and Dirac Equation

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Chapter 1

Introduction

Geometric Algebra is the study of the Clifford algebras, along with their geometric structure, namely the concepts of vector, bivector, area, volume, magnitude (length) and other constructions, such as inner (scalar) products and exterior products.

In this thesis, we study the algebraic foundations of the Clifford algebras, first introduced by William Kingdon Clifford (1845-1879), in 1876-1878, following the work of Hermann Grassmann, who introduced the exterior algebras in 1844.

In Chapter 2, after a brief recollection of the basic properties of vector spaces, we introduce some geometric considerations, in connection with the usual Euclidean $\mathbb{R}^3$ space, namely the inner (scalar) product and the exterior product, which will later be detailed in the case of cross (vector) product.

After the basic presentation of concepts used in vector spaces, we make a step forward and consider the mere product of vectors, thus letting the vector group to be a *ring*. This will allow us to introduce another type of algebraic structures, namely the *algebras*. Just like vector spaces, they can be decomposed in direct sums. Moreover, due to their additional property of possessing a well-defined multiplication of vectors, if the decomposition in direct summands has another key-property (which uses just the difference between vector spaces and algebras, i.e. the product of two vectors), then the decomposition is called *grading*. We will mostly use $\mathbb{Z}_2$-gradings, since these are the ones that appear in the case of Clifford algebras. This particular kind of decomposition will be the topic of Chapter 3.

In Chapter 4, we start by making some considerations about the product of two vectors in the canonical basis of $\mathbb{R}^3$, e_ie_j. We will thus see that this element cannot be a vector or a scalar and it will be called *bivector*. This is the key-step in the construction of the Clifford algebra $\mathfrak{Cl}_2$. Furthermore, a matrix representation of the algebra will be attempted and the concepts of *spinor* and *pinor* groups will be introduced.

In the following chapter, number 5, we will look at exterior (Grassmann) algebras, to understand the starting point for the construction of Clifford algebras, a step that Clifford himself has considered when he introduced the structures that bear his name. After this, we will look at decompositions of the Clifford algebra $\mathfrak{Cl}_3$ over $\mathbb{R}^3$ and of the exterior algebra $\bigwedge \mathbb{R}^3$ in odd and even parts. This decompositions will allow us to *grade* the algebras, over $\mathbb{Z}_2$, as announced before.

Then, in Chapter 6, we will be concerned with the quantum mechanical applications of the Clifford algebras. After a recollection of the time-dependent Schrodinger equation (TDSE) and the Stern-Gerlach experiment, we will present the corrections that Pauli has made to the TDSE, in order to take into account the intrinsic angular momentum of the electron, its *spin* as it is known. Pauli introduced the now-called *Pauli matrices* in order to make the TDSE consistent with the concept of spin. Dirac made a further refinement of this Schrodinger-Pauli equation, following the improvement of Klein-Gordon. In his equation, Dirac not only included the spin of the electron, but also relativistic considerations. He introduced four *Dirac matrices* $\gamma_0, \gamma_1, \gamma_2, \gamma_3 \in \mathcal{M}_4(\mathbb{C})$ that made his equation linear.

In the last section of the last chapter, we see that not only do the Dirac matrices span the space $\mathcal{M}_4(\mathbb{C})$, which is isomorphic to $\mathbb{R}^4$, but also, in some particular combinations involving the Pauli matrices, they span some parts of the exterior algebra $\bigwedge \mathbb{R}^4$ and thus the Clifford algebra $\mathfrak{Cl}_4$ over $\mathbb{R}^4$. This spanning will be detailed further in the second section of the Appendix, whilst the first section gives a further insight in the theoretical physics development of the Dirac equation.

After this final step, we see how powerfull and present are the Clifford algebras in mathematics and quantum mechanics and conclude that the techniques and concepts presented briefly in this work can be applied in all of the realms where one encounters exterior and Clifford algebras. The importance of the Clifford algebras and Geometric Algebra to particle (modern) physics was mainly realized by Dr. David Hestenes, who used extensively Clifford algebra techniques in his particle physics works staring with 1968.

Chapter 2

Vector Spaces

2.1 Definition, Basis, Dimension

This chapter provides the basic recall of the algebraic structure of a vector space. Notions like *linear combinations*, *span*, *generators*, *basis* and *dimension* will be reviewed in the first section. Further, the *mappings* between vector spaces will be introduced in Section 2, which will allow us to construct *bilinear* and *quadratic forms* in Section 4. These notions will help us present the *tensor products* in Section 5 and then give the first geometric flavour of the structures with the presentation *inner* and *exterior* vector products, in Section 7.

In general, the common notion of a vector is considered the oriented segment, with a fixed origin. Formally speaking, this notion is consistent only in some very particular cases, namely two-dimensional vector spaces, which maybe are endowed with some other constructions such as scalar products, distances and others.

We will further give the general definition of a vector space over a commutative field. But for that, let us recall the basic constructions of groups and fields.

Definition 2.1.1 *Let G be a nonempty set and $\cdot : G \times G \to G$ a binary operation. The pair $(G, \cdot)$ is called a* **group** *if it satisfies the following axioms:*

- *"$\cdot$" is an internal law, i.e. $\forall\ x, y \in G, x \cdot y \in G$;*
- *"$\cdot$" is associative, in the sense of $\forall\ x, y, z \in G, (x \cdot y) \cdot z = x \cdot (y \cdot z)$;*

- *There exists a neutral element,* $e \in G$ *such that* $e \cdot x = x \cdot e = x$, $\forall\, x \in G$*;*

- *Any element is invertible with respect to the operation, i.e.* $\forall\, x \in G$, $\exists\, x' \in G$ *such that* $x \cdot x' = x' \cdot x = e$*, the neutral element.*

A further construction, which allows two binary operations, is that of a *ring*. More restrictive conditions for rings give rise to *fields*, which we define below:

Definition 2.1.2 *Let* K *be a nonempty set and* $+ : K \times K \to K$ *(addition) and* $\cdot : K \times K \to K$ *(multiplication) two binary operations. The triplet* $(K, +, \cdot)$ *is called a* **field** *if both* $(K,+)$ *and* $(K, \cdot)$ *are groups.*

When there is no risk of confusion, the multiplication $a \cdot b$ will be simply denoted ab. Also, the neutral element with respect to an additive operation is commonly denoted by 0 (zero) and the one with respect to multiplication, by 1 (one). With that notation, the inverse of an element a with respect to addition can be denoted by $(-a)$, whereas with respect to multiplication, by a^{-1}.

For a group, if the operation is commutative, i.e. $x+y = y+x$, $\forall\, x, y \in G$, then the group itself is called commutative (or abelian, to honor the Norwegian mathematician Niels Henrik Abel, one of the founders of group theory). Moreover, for a field it is required that the additive group to be abelian and if the multiplicative one is also abelian, then the field is called abelian.

The ring structure is more loose, namely it doesn't necessarily require the existence of 1 and of a^{-1} for all elements.

Remark 2.1.1 *In all of the following, all fields will be considered commutative.*

A connection of the two notions, of groups and fields, respectively, is done in the construction of vector spaces. These are a bit more complicated structures and we will define them as follows:

Definition 2.1.3 *Let* K *be a field (called the scalar field) and* $\mathbf{V}$ *a nonempty set, endowed with an additive internal operation* $+ : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ *and an external multiplication* $\cdot : K \times \mathbf{V} \to \mathbf{V}$ *such that the following axioms hold:*

1. *($\mathbf{V}$,+) is an abelian group, called the vector group;*

2. $\forall\ \alpha, \beta \in K$ *and* $\mathbf{x} \in \mathbf{V},\ (\alpha+\beta)\cdot\mathbf{x} = \alpha\cdot\mathbf{x} + \beta\cdot\mathbf{x}$ *;*

3. $\forall \alpha \in K$ *and* $\mathbf{x}, \mathbf{y} \in \mathbf{V},\ \alpha\cdot(\mathbf{x}+\mathbf{y}) = \alpha\cdot\mathbf{x} + \beta\cdot\mathbf{y}$*;*

4. $\forall \alpha, \beta \in K$ *and* $\mathbf{x} \in \mathbf{V}, (\alpha\beta)\cdot\mathbf{x} = \alpha\cdot(\beta\cdot\mathbf{x})$*;*

5. $\forall \mathbf{x} \in \mathbf{V}, 1\cdot\mathbf{x} = \mathbf{x}$*, where* 1 *is the neutral multiplicative element in the field K.*

Then $\mathbf{V}$ *is called a* **K-vector space***, or a vector space over the (scalar) field K.*

Remark 2.1.2 The short notation for a K-vector space is simply ${}_KV$. Also, one may have noticed that the group V along with its elements are written in boldface. It is a convention that applies to vectors, but for the ease of both, reading and writing, we will drop the convention when there is no risk of confusion. Moreover, a convention that we will keep further is that the elements in the field K (named scalars) are denoted with Greek letters, whereas the vectors in V, with latin letters.

Proposition 2.1.1 *The following relations hold, for any* ${}_KV$*:*
1. $0\cdot x = 0_V$*, where the first 0 is from K.*
2. $(-1)\cdot x = -x$*, where -1 is the opposite of 1 in K.*

Proofs 1. From the second axiom in the definition of ${}_KV$, taking $\alpha = \beta = 0$ we get that $(0+0)\cdot x = 0\cdot x + 0\cdot x$. But, of course, 0=0+0, so $0\cdot x = 0\cdot x + 0\cdot x$, which happens for any arbitrary x. Therefore $x = 0$, and since it is a vector, it will be denoted by 0_V.

2. Applying again the second axiom, now for $\alpha = 1$ and $\beta = -1$, we get that $[1+(-1)]\cdot x = 1\cdot x + (-1)\cdot x$, so $0\cdot x = x + (-1)\cdot x \Rightarrow 0_V = x + (-1)\cdot x$. Hence $(-1)\cdot x = -x$. ⊠

Example 2.1.1 The most common example of vector spaces are the spaces ${}_KK^n$, where K is any field and K^n is the cartesian product of K with itself n times. An element x in the field K^n will be said to have n components, namely $x = (x_1, x_2, \cdots, x_n)$ which are informally called its *coordinates*, to make a correspondence with the planar geometry.

The field K is commonly used as $\mathbb{R}$, the field of real numbers or $\mathbb{C}$, the field of complex numbers. Both the addition of vectors and the multiplication with scalars will be defined componentwise, i.e. $(x_1, \cdots, x_n) + (y_1, \cdots, y_n) = (x_1 + y_1, \cdots, x_n + y_n)$ and $\alpha(x_1, \cdots, x_n) = (\alpha x_1, \cdots, \alpha x_n)$.

Subunits of vector spaces will be called vector subspaces, defined below:

Definition 2.1.4 *Let V be a K-vector space and $W \subset V$ a nonempty subset. If W is also a K-vector space, then W is called a* **vector subspace** *of V.*

A property that reunites all the axioms in the definition of a vector space is the one of linearity. This is the reason why sometimes vector spaces are called *linear* spaces. A linear combination of vectors with scalars is the sum $\sum_i \alpha_i x_i$. So we have the following characterisation:

Proposition 2.1.2 *A subset W of a K-vector space V is a K-subspace if it is closed under linear combinations, i.e.* $\forall\ x, y \in W$ *and* $\alpha, \beta \in K$, *we have that* $\alpha x + \beta y \in W$.

One can easily check that, indeed, this property is equivalent to all the axioms in the definition of a vector space.

Definition 2.1.5 *Let ${}_KV$ and $S = \{v_1, \cdots, v_m\} \subset V$. S is called a* **system of generators** *for V if all the linear combinations of scalars with (not necessarily all) vectors in S form the entire space V. The notation is sometimes V=sp(S), to mean that V is spanned by S.*

Definition 2.1.6 *Let S be a subset of m vectors from ${}_KV$. S is called* **linearly independent** *if for all $\alpha_1, \cdots, \alpha_m \in K$, the equality $\sum_{i=1}^{m} \alpha_i x_i = 0$ implies that $\alpha_i = 0, \forall i = 1, 2, \cdots, m$.*

Otherwise, S is called **linearly dependent***.*

Now we can extract the "core" of any vector space, the fundamental constituents of it, namely its base.

Definition 2.1.7 *Let V be a K-vector space and S a system of vectors. If S is both a system of generators and a linearly independent set, then it is called the* **base** *of V. Its number of elements will be called the* **dimension** *of V, denoted $dim_K V$.*

Bases are very important when dealing with vector spaces, because the definitions above impliy that *any vector in V can be uniquely written as a linear combination of vectors in the base with scalars in K.*

Example 2.1.2 In the vector space ${}_K K^n$ presented above it can be easily checked that its base has n elements, denoted usually by e_i, $i = 1, \cdots, n$, which have a simple form, namely all the components ("coordinates") of e_i are zero, except for the i-th one, for all i from 1 to n. It can be easily checked that this set is indeed a basis.

Remark 2.1.3 *The basis of a vector space needs not be unique. But what can be proven is that all the bases of a vector space have the same cardinality, namely the dimension of the vector space.*

We now give an useful construction for changing bases of a vector space. Consider we have two bases, B and B' and we know the components of all the vectors in the space with respect to both bases, i.e. we know the scalars that appear in the decomposition of any vector in a linear combination of vectors in the base with scalars in the field. Suppose $dim_K V = n$ so that we will have exactly n components for all vectors (and thus $cardB = cardB' = n$).

Put all the elements in the base B in a $n \times n$ matrix, denote it $\mathbb{B}$, and let each set of coordinates to be a line (or a row) in the matrix. Do the same for the elements in B' (to get $\mathbb{B}'$) and then write the matrix equation $\mathbb{B} = A\mathbb{B}'$ and solve for A. The solution will be the matrix that makes the "passing" from the base B to B'.

If one knows the components of a vector x with respect to the base B, write them as a column matrix and multiply it with A to get the components with respect to B'.

Remark 2.1.4 *Since determining the matrix A means to solve a system of n equations with n unknowns which come from bases of the vector space, it follows that the solution is unique and it implies that the matrix A is invertible, so the passing can be made in both directions.*

Definition 2.1.8 *Let V and W be two K-vector spaces. Their* **sum** *is $V + W = \{v + w \mid v \in V, w \in W\}$, i.e. the set containing all the sums of vectors, one from V and the other from W.*

The sum is called **direct sum**, *written as $V \oplus W$ iff $V \cap W = \{0\}$.*

Direct sum of vector spaces have very important properties, which make them essential in decomposing vector spaces or other structures (direct sums of any algebraic structures are defined similarly). Namely, we have the following:

Theorem 2.1.1 *Let ${}_K V$ and ${}_K W$ be vector spaces and denote $S = V \oplus W$. Then for any s in S, there exist $v \in V$ and $w \in W$ such that $s = v + w$, decomposition which is unique.*

Proof The first part of the theorem tells nothing new, because the decomposition $s = v + w$ can be found in any sum $V + W$, even if it is not direct. The most important fact is the remark that *decomposition is unique.* Let us see how this comes out.

If the sum $V \oplus W$ is direct, from the definition, we have that $V \cap W = \{0\}$. Suppose, further, that $s \in S$ has two different decompositions, $s = v + w = v' + w'$, with every vector from its corresponding space. Then $v - v' = w' - w$. If we denote by x this vector, it must be in $V \cap W$, because it can be written both using vectors in V and vectors in W. It then follows that $x = 0$ and, as a consequence, $v = v'$ $w = w'$, which make the decomposition of s unique. ⊠

2.2 Linear Maps

Definition 2.2.1 *Let V and W be two vector spaces over the same scalar field K and $f : V \to W$ be a function (map). f is called a* **linear map** *(or, more formally, a vector space homomorphism) iff:*

- $f(x + y) = f(x) + f(y),\ \forall x, y \in V$
- $f(\lambda x) = \lambda f(x),\ \forall \lambda \in K$ *and* $x \in V$

Equivalently, $\forall x, y \in V$ *and* $\alpha, \beta \in K,\ f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$.

The reason for which this kind of map is called a linear map is that it preserves linear combinations, i.e. $f(\sum_i \alpha_i x_i) = \sum_i \alpha_i f(x_i)$.

Remark 2.2.1 *An immediate consequence when taking $\lambda = 0_K$ in the definition and using Proposition 2.1.1 is that $f(0_v) = 0_W$.*

When dealing with homomorphisms (be them group, ring, field or vector space homomorphisms), algebraists like to say that one must "salivate" like Pavlov's dog and ask for two objects that provide further insight to the structure and properties of homomorphisms, namely its kernel and image, defined below.

Definition 2.2.2 *Let f be a $_KV \to_K W$ homomorphism. Its kernel, denoted by $Kerf$ and image, denoted by Imf are defined as follows:*

$$Kerf = \{x \in V \mid f(x) = 0_W\}$$
$$Imf = \{y \in W \mid \exists x \in V \text{ such that } f(x) = y\}$$

One can easily prove that $Kerf$ is a vector subspace of V and Imf a vector subspace of W.

If the map f is bijective, then it is called an *isomorphism* of vector spaces and the spaces will be called isomorphic, denoted $V \approx W$.

An useful construction when dealing with vector spaces is their *dual*, defined as follows:

Definition 2.2.3 *Let V be a K-vector space. The set $V^* = \{f \mid f : V \to K,\ f \text{ is a linear map}\}$ is called the* **dual space** *of V.*

But this construction would be almost useless if there was not the following theorem:

Theorem 2.2.1 *The dual space V^* of a vector space V is itself a vector space isomorphic to V.*

Proof The first step is to make $(V^*, +)$ an abelian group. Define the addition of two elements in V^* pointwise, i.e. $(f + g)(x) = f(x) + g(x),\ \forall f, g \in V^*$. This addition is internal, due to the powerful structure of V and K between which f and g make correspondence. It is also associative, like any addition, the neutral element will be the zero-map, i.e. the one that sends any vector to the zero scalar and the inverse of any map f will be $-f$, defined by $(-f)(x) = -f(x) = f(-x)$, due to linearity. Thus, V^* becomes an additive abelian group.

Next, we have to define the external operation, namely the multiplication with scalars. This is natural; we will define αf to be $\alpha f(x) = f(\alpha x)$, for all $f \in V^*,\ \alpha \in K,\ x \in V$.

Thus, we have made V^* a K-vector space.

Now let $B = \{e_1, \cdots, e_n\}$ be a basis of V. Define the following set: $B^* = \{e_1^*, \cdots, e_n^*\} \subset V^*$ like this:

$$e_i^* : V \to K,\ e_i^*(e_j) = \delta_{ij}\ (\text{the Kronecker symbol}),\ \forall i, j = 1, 2, \cdots, n.$$

Using the fact that any vector in V can be written using all the elements in B, we can extend by linearity the definition of e_i^* for any vector in V. It is an immediate check to prove that B^* defined so is a basis for V^* and since it has the same cardinality as B, it follows that the two spaces are isomorphic. ⊠

The basis B^* constructed above is called the *dual base* of B.

When we have defined some linear map on two vector spaces, an useful construction provides a connection with square matrices. This construction is called *the matrix associated to the linear map in a fixed base.*

To obtain this matrix, we must proceed in a very similar way with the construction of the matrix used for passing from one base to another.

So, let $f : V \to W$ be a linear map and $B_V = \{v_1, \cdots, v_n\}$ be a basis for V and $B_W = \{w_1, \cdots, w_m\}$ be a base for W. For all $i = 1, 2, \cdots, n$, the vector $f(v_i) \in W$ and so it is a linear combination of vectors in the set B_W. Therefore, $\exists \alpha_{ij} \in K,\ i, j = 1, 2, \cdots, n$ such that $f(v_i) = \sum_j \alpha_i j w_j$.

The set (a_{ij}) defines an $n \times m$ matrix, which is the construction that we needed.

2.3 Eigenvalues and Eigenvectors

Definition 2.3.1 *Let f be a linear map from a K-vector space V to itself. If there exists a scalar $\lambda \in K$ such that $f(x) = \lambda x$, then x is called an* **eigenvector** *for f and λ its eigenvalue.*

Remark 2.3.1 *It is an easy exercise to prove that the eigenvectors of a linear map together with the zero vector form a vector subspace for the space V.*

We will give further an algorithm to find the eigenvectors and their eigenvalues for any vector space V.

Suppose $dim_K V = n$ and that $B = (e_i),\ i = 1, 2, \cdots, n$ is a basis for V. Let x be a vector in V and $x_i, i = 1, 2, \cdots, n$ be its components with respect to the basis B.

If x is an eigenvector, then there exists a scalar λ in K such that $f(x) = \lambda x$ or, written by components,

$$f(\sum_i x_i e_i) = \lambda \sum_i x_i f(e_i).$$

Let $A = (a_{ij}) \in M_n(K)$ be the matrix associated to f. Then expanding the relation above, we get, by components, that

$$(a_{ij} - \lambda\delta_{ij})x_j = 0,\ \forall i = 1, 2, \cdots, n.\ (*)$$

This relation can be put in an elegant form using determinants, i.e.

$$det(A - \lambda I_n) = 0,\ I_n = (\delta_{ij}),\ i, j = 1, 2 \cdots, n \text{ being the unit matrix.}$$

This equation is sometimes presented in a polynomial form, namely $P(\lambda) = det(A - \lambda I_n)$ and it is called the *characteristic polynomial of f.* Thus we search for its roots. After finding a root, say, λ_0, we can replace it in the system (*) to get a solution $(x_1, \cdots, x_n)$ which will give the components of the eigenvector corresponding to the eigenvalue λ_0.

2.4 Bilinear and Quadratic Forms

Definition 2.4.1 *Let V be a K-vectors space and $f : V \times V \to K$ a function such that:*

- $f(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 f(x_1, y) + \alpha_2 f(x_2, y);$
- $f(x, \beta_1 y_1 + \beta_2 y_2) = \beta_1 f(x, y_1) + \beta_2 f(x, y_2)$

$\forall x, y_1, y_2 \in V,\ \beta_1, \beta_2 \in K.$

Then f is called a **bilinear form** *defined on V.*

If we also have that $f(x, y) = f(y, x),\ \forall x, y \in V$, then the form is callled *symmetric.*

The matrix $A = (f(e_i, e_j)), i, j = 1, 2, \cdots, n$ is called the matrix associated to the bilinear form f if we have fixed a basis $B = (e_i), i = 1, \cdots, n$ for an n-dimensional vector space V.

Definition 2.4.2 *Let $Q : V \to K$ be a map such that there exists a bilinear symmetric form $f : V \times V \to K$ such that f(x,x)=Q(x), for all x in V. Then Q is called a* **quadratic form** *associated to the bilinear form f.*

2.5 Tensor Products

Making tensor products of vector spaces is a way of obtaining new algebraic structures from old ones. But as we are about to see, the construction is highly counter-intuitive and it may seem difficult. Before presenting the formal definition, let us tell their "story".

Let V and W be vector spaces over the same field K, with $dim_K V = n$ and $dim_K W = m$. Let $(e_i)_i$ and $(f_j)_j$ be their bases. The vector space $T = V \otimes W$ is called their tensor product and it will be a K-vector space with $dim_K T = mn$.

If $x = \sum_i x_i e_i$ and $y = \sum_j y_j f_j$ are the decompositions of two arbitrary vectors in V and W, then their corresponding vector in T will be

$$t = v \otimes w = \sum_i \sum_j x_i y_j e_i \otimes f_j.$$

If we are to consider a *tensor map*, $\tau : V \times W \to V \otimes W$, then it would be bilinear, namely

$$\begin{aligned} \tau(x + x', y) = \tau(x, y) + \tau(x', y) &\Leftrightarrow (x + x') \otimes y = (x \otimes y) + (x' \otimes y) \\ \tau(x, y + y') = \tau(x, y) + \tau(x, y') &\Leftrightarrow x \otimes (y + y') = (x \otimes y) + (x \otimes y') \\ \tau(\lambda_1 x, \lambda_2 y) = \lambda_1 \lambda_2 \tau(x, y) &\Leftrightarrow (\lambda_1 x) \otimes (\lambda_2 y) = (\lambda_1 \lambda_2)(x \otimes y). \end{aligned}$$

Moreover, if we take any bilinear map, then it factors uniquely through the tensor product. Let $\beta : V \times W \to T$ be a bilinear map between the two vector spaces. Take any vector $(x, y) \in V \times W$, which has the basis, say, (e_i, f_j). Then, by bilinearity, we have that

$$\beta(x, y) = \beta(\sum_i x_i e_i, \sum_j y_j f_j) = \sum_i \sum_j (x_i y_j \beta(e_i, f_j)).$$

We thus see the correspondence of bases, which extends, by linearity, to all elements in the vector spaces.

If we take now the linear map, $\theta : V \otimes W \to T$, it will send $e_i \otimes f_j$ to $\beta(e_i, f_j)$ and thus $x \otimes y$ to $\beta(x, y)$. In fact, θ is the only linear map such that $\beta = \theta \circ \tau$. That is what we mean that any bilinear map factors uniquely through the tensor product.

Before we can present the formal definition of the tensor product, another concept is required.

Definition 2.5.1 *Let* $\beta :_K V \times_K W \to_K X$ *be a vector space homomorphism.* β *is called a* **bihomomorphism** *iff* $\forall v, v' \in W, w, w' \in W, \alpha \in K$,

- β *is biadditive, i.e.*
  - $\beta(v + v', w) = \beta(v, w) + \beta(v', w)$;
  - $\beta(v, w + w') = \beta(v, w) + \beta(v, w')$;
- β *is balanced, i.e.* $\beta(\alpha a, b) = \beta(a, \alpha b)$.

Now we can formally define tensor products of vector spaces.

Definition 2.5.2 *Let* A, B *be* K*-vector spaces. The* **tensor product** *of* A *and* B*, denoted* $A \otimes B$ *is an abelian group, together with a tensor map* $\tau : A \times B \to A \otimes B$*, which is also a bihomomorphism. Moreover, for any other abelian group* C *and bihomomorphism* $\psi : A \times B \to C$*, there exists a unique homomorphism of abelian groups,* $\phi : A \otimes B \to C$*, such that* $\psi = \phi \circ \tau$.

Remark 2.5.1 Tensor products can also be taken considering modules or algebras. From the construction in the begining of the section, we see that the tensor product of two vector spaces is unique. Moreover, it exists (it can be constructed) for any two structures (modules, vector spaces or algebras).

2.6 The Complexified Vector Space

Definition 2.6.1 *Let* V *be an* $\mathbb{R}$*-vector space. Define the following operations on* $V \times V$:

- $(x, y) + (x', y') = (x + y, x' + y'),\ \forall (x, y), (x', y') \in V \times V$
- *Let* $\alpha = a + bi \in \mathbb{C}$*. Then define* $(a + bi)(x, y) = (ax - by, ay + bx),\ \forall (x, y) \in V \times V$

With these operations, we have made $V \times V$ *a* $\mathbb{C}$*-vector space, called the* **complexified space** *of* V.

Remark 2.6.1 $dim_{\mathbb{R}} V = dim_{\mathbb{C}} V \times V$*, because if* $(e_i)_i$ *is a basis for* V*, then* $((e_i, 0))_i$ *is a basis for* $V \times V$.

2.7 Inner and Exterior Products

Due to the formal construction of vector spaces, one can only "combine" them with scalars, to get linear combinations or rescaling. But no multiplication of vectors is defined. In this section, we will present two kinds of vector products introduced to further operate with vectors, namely the inner (scalar) product and the exterior (vector) product.

2.7.1 Inner Product

Inner (scalar) products were first introduced in order to "measure" arbitrary vectors, i.e. to define their norm (length). The formal definition of an inner product is as follows:

Definition 2.7.1 *Let V and W be two K-vector spaces. The map $< \cdot, \cdot >$: $V \times W \to K$ is called an* **inner product** *if it is:*

1. *Positive, i.e. $< x, x > \geq 0, \forall x \in V$ (if $K = \mathbb{C}$, this means that $< x, x >$ is real and nonnegative);*

2. *Definite, i.e $< x, x >= 0 \Leftrightarrow x = 0$;*

3. *Additive in first slot, $< x + x', y >=< x, y > + < x', y >$, $\forall x, x' \in V, y \in W$;*

4. *Homogeneous in first slot, $< \alpha x, y >= \alpha < x, y >$, $\forall x \in V, y \in W, \alpha \in K$;*

5. *Conjugate symmetric, $< x, y >= \overline{< y, x >}$, $\forall (x, y) \in V \times W$, where $\overline{z}$ means the complex conjugate of z.*

The reason for which the inner product is sometimes called *scalar* product is that it takes a vector (or a pair of vectors, more precisely) to a scalar, since the map takes its values in the field of scalars.

Example 2.7.1 The most common example of an inner product is the so-called **Euclidean product**. Let $V = W =_K K^n$ be the vector space that we consider and define the inner product as follows:

$$< (x_1, \cdots, x_n), (y_1, \cdots, y_n) >= \sum_i x_i \overline{y_i}.$$

It is an easy computation to check that this map respects the conditions set in the definition.

The reason why this map is called *Euclidean* product is that if we take $K = \mathbb{R}$ and $n = 2$ or $n = 3$, we get a very common vector space, the real plane for $n = 2$ or the real space, for $n = 3$. In this space we can apply considerations of basic Euclidean geometry. Moreover, a geometric interpretation can be given for this product, if one also defines a norm first (which will give the "length" of a vector).

Definition 2.7.2 *Let V be a vector space and $< \cdot, \cdot >$ be an inner product defined on V (the pair $(V, < \cdot, \cdot >)$ is called an inner product space). Let x be a vector in V. The* **norm** *of x, denoted by $\| x \|$, is defined as:*

$$\| x \| = \sqrt{< x, x >}$$

Now, if we apply this notion to our Euclidean product (to get a so-called *Euclidean space*, which consists of the space $\mathbb{R}^n$, together with the Euclidean inner product), then we get the well-known expression for the norm of a vector $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, namely $\| x \| = \sqrt{\sum_i x_i^2}$.

An even more common geometrical interpretation can be made if one takes the space $\mathbb{R}^2$. Then the inner product of two vectors $a, b \in \mathbb{R}^2$ will be

$$< a, b > = a \cdot b = \| a \| \cdot \| b \| \cdot cos(\widehat{a, b}), \text{ with}$$
$$\| a \| = \sqrt{a_1^2 + a_2^2}, \text{ its coordinates.}$$

2.7.2 Exterior Products

Just like in the case of an inner product, there is another way of "combining" two (or more) vectors, namely by making their exterior product.

Definition 2.7.3 *Let V, W be K-vector spaces. A map $\wedge : \underbrace{V \times V \cdots \times V}_{n\ times} \to W$ is called an* **exterior** *product iff it is:*

1. *Alternating, which means that $\wedge(v_1, \cdots, v_n) = 0$ ($\forall v_i \in V$) whenever there exist i, j such that $v_i = v_j$;*

2. *Skew-symmetric, i.e. $\wedge(v_1, \cdots, v_n) = - \wedge (v_n, \cdots, v_1)$, $\forall v_i \in V$;*

3. *$\forall v_1, \cdots, v_n \in V$, $< \wedge(v_1, \cdots, v_n), v_i >= 0, \forall i = 1, 2, \cdots n$, where $< \cdot, \cdot >$ is the inner product.*

Remark 2.7.1 The exterior product is sometimes called *vector* product, because the result of it is another vector. Moreover, the product is "exterior" in the sense that its result is not necessarily in the same space as the factors.

From the third property of the exterior product, one can see that it produces, in fact, a vector which is orthogonal to any of the vectors involved in the product. Therefore, it must be in the (hyper)plane that is orthogonal to the respective vector.

The concept of an exterior (wedge) product will be clearer after introducing bivectors and Clifford algebras, the topic of Chapter 4.

Chapter 3

Graded Algebras

3.1 Algebras

Algebras are the next structures that we will present. Informally speaking, algebras are structures in which allmost any elementary operation (addition, subtraction, multiplication, "division" = multiplication with inverse) can be performed between any two elements. It is less restrictive than a vector space; thus, a vector space is, in fact, a group (of vectors) over a field (of scalars) and an algebra is a ring (still consisting of vectors) considered over a field (of scalars). Moreover, instead of vector spaces, one can consider groups over rings, which will be caled *modules* while rings over rings are still called algebras. Let us define the concept properly:

Definition 3.1.1 *Let R and A be rings. A is called an* **R-algebra** *(or algebra over the ring R) if it has the structure of an R-module (or, more generally, an R-vector space, if R is a field), and the operations are compatible, namely:*

1. $a(b+c) = ab + ac$ *and* $(a+b)c = ac + bc,\ \forall a, b, c \in A$;
2. $(ra)b = a(rb) = r(ab),\ \forall a, b, c \in A$ *and* $r \in R$.

Moreover, it is required that the ring A is unital, i.e. has a multiplicative identity 1.

Algebras defined like this are also called *associative* algebras, because of their second property. If that would not hold, the algebra is called *nonassociative*. In this work we will mainly deal with associative algebras.

- Basic examples of $\mathbb{R}$-algebras are $\mathbb{C}$ and $\mathbb{H}$, the ring of complex numbers and the quaternion ring.
- Every ring is a $\mathbb{Z}$-algebra, via the equality $\underbrace{x + x + \cdots + x}_{\text{n times}} = n \cdot x$, for any integer n.
- The ring of square $n \times n$ matrices, $M_n(R)$ with entries from the ring R is an R-algebra

Note that the structure of an algebra allows vector multiplication, because the vectors form a ring, not just a group. It is important to point out that the product of two vectors has nothing to do with inner or outer products, but instead, it is a simple "multiplication" of elements in a ring.

The following definitions are natural extrapolations of the similar notions for the other algebraic structures:

Definition 3.1.2 *A subalgebra S of an R-algebra A is a nonempty subset of A such that S itself is an R-algebra. An algebra homomorphism is a ring homomorphism that is also a vector space homomorphism (or module homomorphism), namely a linear ring homomorphism.*

3.2 Grading

The operation of grading of an algebra is merely a decomposition of it, as the following definition states:

Definition 3.2.1 *Let A be a K-algebra. Regard A as a K-vector space. A is called* **G-graded** *(with G a commutative group) iff there exist vector spaces $A_g,\ g \in G$ such that $A = \bigoplus_{g \in G} A_g$ and $A_{g_1} \cdot A_{g_2} \subset A_{g_1+g_2},\ \forall g_1, g_2 \in G$.*

Remark 3.2.1 The set $A_{g_1} \cdot A_{g_2}$ consists of all elements of the form $a \cdot b$, such that $a \in A_{g_1}$ and $b \in A_{g_2}$.

Note that the definition allows G to be any commutative group, but if we take $G = \mathbb{Z}$, the additive group of integers, the conditions in the definition become friendlier, because we will have that $A = \bigoplus_{i=0}^{\infty} A_i$, and $A_i \cdot A_j \subset A_{i+j}$.

Example 3.2.1 An example of grading easy to understand is the grading of the polynomial algebra. Let $K[X]$ be the algebra of polynomials with coefficients in K, taken over the same field K. An immediate grading is:

$$K[X] = \bigoplus_i K_i[X],$$

where the vector spaces $K_i[X]$ are formed by multiples of terms of the form αx^i, with $\alpha \in K$.

The following property is useful and provides another element that characterises grading. For simplicity of notation, let $G = \mathbb{Z}$.

Proposition 3.2.1 *Let A be a $\mathbb{Z}$-graded algebra and 1_A be its neutral element with respect to the ring structure. Then $1_A \in A_0$.*

Proof From the decomposition that appears in the definition of the grading, we have that:

$$1 = \sum_{i=0}^{\infty} a_i, \text{ with } a_i \in A_i.$$

Let $x \in A_0$. Because $x \cdot 1 = x$, we get from the above relation, by multiplication with x:

$$x = \sum_{i=1}^{\infty} x a_i, \text{ with } x a_0 \in A_0,\ x a_i \in A_{i+0} = A_i,$$

for all i, from the second property in the definition of grading.

But because $x \in A_0$ and the sum of vector spaces is direct, it follows from **Theorem 2.1.1** that $xa_1 = xa_2 = \cdots = 0$ and that $x = xa_0$. But because x is an arbitrary element, it follows that $a_0 = 1_A$, for the relation cannot hold for all x. ⊠

Chapter 4

Bivectors and Clifford Algebras

4.1 Bivectors

Clifford algebras are generalizations of the common associative (real) algebras, namely the real algebra $\mathbb{R}$, the complex algebra $\mathbb{C}$ and the algebra of quaternions, $\mathbb{H}$.

For simplicity and to suit the main purpose of this work, which is to provide connections between Clifford algebras and the Dirac equation, we will treat with priority the algebras $\mathfrak{Cl}_0 \approx \mathbb{R}$, $\mathfrak{Cl}_1 \approx \mathbb{C}$ and $\mathfrak{Cl}_2 \approx \mathbb{H}$.

For the construction of these algebras, let us start with the basic vector space $\mathbb{R}^2$, endowed with the Euclidean inner product. The canonical base of the space is $\{e_1, e_2\}$, with $e_1 = (1,0)$ and $e_2 = (0,1)$. The length of a vector $r = xe_1 + ye_2$ is $|r| = \sqrt{x^2 + y^2}$. If we take the scalar product of a vector with itself, it will give the square of its length (norm). Now we want to introduce a product of vectors in such a way that

$$(xe_1 + ye_2)^2 = x^2 + y^2.$$

What properties will this product have? Well, first note that, from a strictly algebraic point of view,

$$(xe_1 + ye_2)^2 = x^2 e_1^2 + y^2 e_2^2 + xy(e_1 e_2 + e_2 e_1).$$

Now, to identify the two relations above, we must require that

$$e_1 e_2 = -e_2 e_1 \text{ and } e_1^2 = e_2^2 = 1,$$

which brings no novelty, because these conditions say that the basis vectors must be of length 1 and orthogonal (in one word, orthonormal). What is new follows from computing the length of the vector (e_1e_2):

$$(e_1e_2)^2 = (e_1e_2)(e_1e_2) = (e_1e_2)(-e_2e_1) = e_1(-1)e_1 = -e_1^2 = -1.$$

It implies that the quantity e_1e_2 is neither a scalar (because we are in $\mathbb{R}$), nor a vector, since it cannot have a negative magnitude. This new kind of object is called a **bivector**.

The structure of a Clifford algebra will be introduced on a space which contains scalars, vectors and bivectors. This new concept makes the generalization of scalar and vector spaces. But first, let us introduce the product which is characteristic to these algebras.

Definition 4.1.1 *Let $a = a_1e_1 + a_2e_2$ and $b = b_1e_1 + b_2e_2$ be two vectors in the real plane. The* **Clifford product** *of a and b is defined like:*

$$ab = a_1b_1 + a_2b_2 + (a_1b_2 - a_2b_1)e_{12},$$

where e_{12} is the bivector e_1e_2.

If we look at the Clifford product with enough attention, we will observe that it is a generalization of the products that we introduced in the second chapter. In fact

$$a_1b_1 + a_2b_2 = a \cdot b, \text{ the scalar product and}$$
$$a_1b_2 - a_2b_1 = a \wedge b = a \times b, \text{ the vector (exterior) product.}$$

This remark ensures that the Clifford product produces a combination of a vector and a scalar.

4.2 The Clifford Algebra $\mathfrak{Cl}_2$

In order to construct the Clifford algebra $\mathfrak{Cl}_2$, we take the objects above and use them as a basis. Concretely, we have the following:

Definition 4.2.1 *The* **Clifford algebra** *$\mathfrak{Cl}_2$ is a vector space spanned by $\{1,\ e_1,\ e_2,\ e_{12}\}$, endowed with the Clifford product of vectors defined, for any two vectors $a, b \in \mathfrak{Cl}_2$ by:*

$$ab = a \cdot b + a \wedge b.$$

The index 2 in denoting the algebra $\mathfrak{Cl}_2$ comes from considering it a generalization of the real plane $\mathbb{R}^2$. Note that $dim\mathfrak{Cl}_2 = 2^2 = 4$ and we will see that, in general, $dim\mathfrak{Cl}_n = 2^n$.

In fact, let us anticipate some results which may seem natural.

Remark 4.2.1 The Clifford algebra $\mathfrak{Cl}_0$ is the real vector space spanned by $\{1\}$, thus it is the scalar field, which has dimension $1 = 2^0$, considered as a space over $\mathbb{R}$, $\mathfrak{Cl}_1$ will be spanned by $\{1,\ e_1\}$ and thus will have a dimension of $2 = 2^1$ and $\mathfrak{Cl}_3$ will be spanned by $\{1,\ e_1,\ e_2,\ e_3,\ e_{12},\ e_{13},\ e_{23},\ e_{123}\}$ and will be of dimension $8 = 2^3$.

Moreover, we have a natural isomorphism $\phi : \mathfrak{Cl}_2 \to \mathbb{C},\ \phi(x + ye) = x + yi$, because any element in $\mathfrak{Cl}_2$ has a scalar (x) and a vector (y) component, e being a generic notation for an element in the basis. We will examine the isomorphism of a part of $\mathfrak{Cl}_3$ with $\mathbb{H}$ when we will discuss in detail the Clifford algebra of dimension 8.

4.3 Decomposition of Vectors

Let us examine further the $\mathfrak{Cl}_2$ algebra to see what are its connections to the real plane. In this section, we will see how an arbitrary vector r can be decomposed upon two nonparallel directions a and b. This means to determine the coefficients that appear in $r = \alpha a + \beta b$.

If we consider the exterior product $r \wedge b$, we get that $r \wedge b = (\alpha a + \beta b) \wedge b = \alpha(a \wedge b)$, where we used the fact that $b \wedge b = 0$. In a similar manner, $r \wedge a = \beta(a \wedge b)$. Thus, we may write:

$$\alpha = \frac{r \wedge a}{a \wedge b} \text{ and } \beta = \frac{r \wedge b}{a \wedge b}.$$

A geometric view of this kind of expressions can be made if one thinks that the wedge product is an oriented area, since $a \wedge b$ is the area of the parallelogram that has vertices a and b.

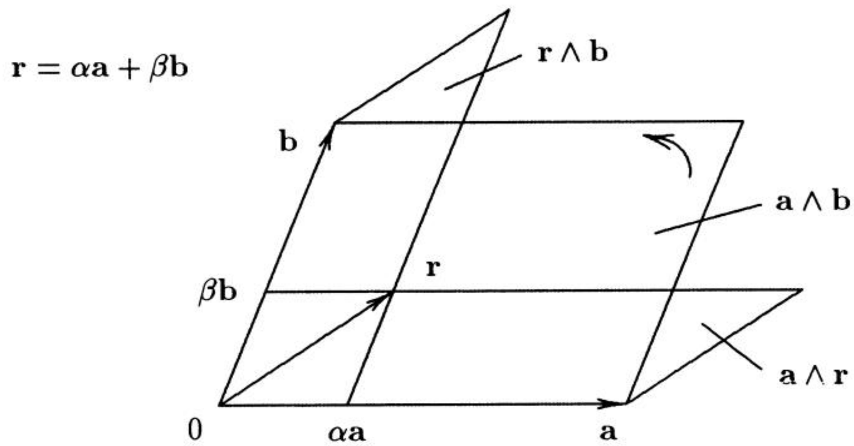

Figure 4.1: The decomposition of a vector in $\mathfrak{Cl}_2$

4.4 Matrix Representation of $\mathfrak{Cl}_2$

The correspondence between $\mathfrak{Cl}_\mathfrak{n}$ and $M_{2^n}(\mathbb{R})$ is a very useful one, firstly because it allows a matrix representation of the algebra, similar to the case of a vector space. Moreover, we will see that for the case of $\mathfrak{Cl}_2$ we will have some matrices, denoted by γ, which are called *Dirac matrices*, that span the space.

Let us start with the simpler algebra, $\mathfrak{Cl}_1$. Let ρ be the representation map, $\rho : \mathfrak{Cl}_\mathfrak{n} \to M_{2^n}(\mathbb{R})$. The natural association is $\rho(1) = I_2$. Then, in order to preserve the relations between the basis elements, namely that any vector has the norm equal to 1 and the bivector has the norm -1, one easily obtains that:

$$\rho(x + ye) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

A generalization leads to the matrix representation of $\mathfrak{Cl}_2$, given by:

$$\rho(1) = I_4$$

$$\rho(e_1) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\rho(e_2) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\rho(e_{12}) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

In physics, the common notations are $\gamma_1 = \rho(1) = I_4,\ \gamma_2 = \rho(e_1),\ \gamma_3 = \rho(e_2),\ \gamma_4 = \rho(e_{12})$ and the corresponding matrices are called *Dirac matrices*, to honor the English physicist P.A.M. Dirac.

Remark 4.4.1 The above representation is not unique, although it is called the *canonic* representation and it is used in most cases. We chose to present it in this work because it is somehow natural and easy to understand. Moreover, the above matrices are the simplest that exhibit all the properties that the Dirac matrices possess (we will detail them in **Chapter 6).**

4.5 Spinor and Pinor Group in $\mathfrak{Cl}_2$

4.5.1 Even and Odd Part of $\mathfrak{Cl}_2$

Just like functions, the Clifford algebra $\mathfrak{Cl}_2$ can be divided into two parts, an odd part and an even part. This algebra contains both the real plane, because $\mathbb{R}^2$ is spanned by $\{e_1,\ e_2\}$, and the complex plane, because one can easily find that the space $\mathbb{C}$ is isomorphic to the space spanned by $\{1,\ e_{12}\}$, since $e_{12}^2 = -1$, just like i, the imaginary unit, and any complex number has a real and an imaginary part and so on.

Moreover, the only common element of the two spaces is 0, which means that the Clifford algebra $\mathfrak{Cl}_2$ can be decomposed as a direct sum, due to **Definition 2.1.8**,

$$\mathfrak{Cl}_2 = {\mathfrak{Cl}_2}^+ \oplus {\mathfrak{Cl}_2}^-.$$

${\mathfrak{Cl}_2}^+$ is called the *even part* of the algebra and is identified with $\mathbb{C}$, whereas ${\mathfrak{Cl}_2}^-$ is called the *odd part* of it, represented by $\mathbb{R}^2$. The names *odd* and *even* are due to the number of vectors appearing in the space. These considerations allow us to make the following remarks:

1. complex number · complex number = complex number;
2. vector · complex number = vector;
3. complex number · vector = vector;
4. vector · vector = complex number.

These results can be interpreted also using the components in the canonical basis, identifying "vectors" with elements in $\mathfrak{Cl}_2$ which are spanned only by e_1 and e_2 and "complex numbers" with those that are spanned by 1 and the bivector e_{12}.

Furthermore, **Theorem 2.1.1** tells us that any vector in the algebra has an "even" ("complex") part and an "odd" ("real") part and that such decompositions are unique in terms of factors.

Let us rewrite the above remarks in symbols, making the numbers correspond:

1. $\mathfrak{Cl}_2{}^+ \cdot \mathfrak{Cl}_2{}^+ \subset \mathfrak{Cl}_2{}^+$
2. $\mathfrak{Cl}_2{}^- \cdot \mathfrak{Cl}_2{}^+ \subset \mathfrak{Cl}_2{}^-$
3. $\mathfrak{Cl}_2{}^+ \cdot \mathfrak{Cl}_2{}^- \subset \mathfrak{Cl}_2{}^-$
4. $\mathfrak{Cl}_2{}^- \cdot \mathfrak{Cl}_2{}^- \subset \mathfrak{Cl}_2{}^+$

If we denote $\mathfrak{Cl}_2{}^+$ with $(\mathfrak{Cl}_2)_0$ and $\mathfrak{Cl}_2{}^-$ with $(\mathfrak{Cl}_2)_1$, all the four relations above can be condensed in

$$(\mathfrak{Cl}_2)_j \cdot (\mathfrak{Cl}_2)_k \subset (\mathfrak{Cl}_2)_{j+k}, \text{ with } j, k \in \mathbb{Z}_2.$$

This fact, along with the direct sum decomposition of the algebra in an even and odd part allow us to say that we have made a $\mathbb{Z}_2$ *grading* of the algebra or an *even-odd grading*.

4.5.2 Spinor Group Spin(2)

Let us consider the plane rotations, in the complex plane $\mathbb{C}$. Let $z = x+iy$ be a complex number, geometrically represented as the oriented segment joining $(0,0)$ with the point (x,y). A counter-clockwise rotation with an angle φ is represented by

$$\mathbb{C}Rot_{\varphi}(z) = (cos\varphi + isin\varphi)z.$$

The factor $cos\varphi + isin\varphi$ is a complex number whose modulus is 1. It is said to belong to the *unitary group*, $U(1) = \{z \in \mathbb{C} \mid z\bar{z} = |z|^2 = 1\}$. This set forms a group with the common multiplication of complex numbers.

Now let us consider the counter-clockwise rotations in the vector plane $\mathbb{R}^2$. Let $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$. Then

$$\mathbb{R}^2 Rot_{\varphi}(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} cos\varphi & -sin\varphi \\ sin\varphi & cos\varphi \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

This time, the rotation matrix $\begin{pmatrix} cos\varphi & -sin\varphi \\ sin\varphi & cos\varphi \end{pmatrix}$ is said to belong to the *special orthogonal group*, $SO(2) = \{R \in M_2(\mathbb{R}) \mid R^T R = 1,\ detR = 1\}$. (It can be shown that $U(1) \approx SO(2)$.)

If we come back to our decomposition of the Clifford algebra into a complex (even) part and a real (odd) part, we identified $i = e_{12}$. Thus, the rotations in the complex plane can be seen via Clifford multiplication, the only modification being that instead of the imaginary unit we will have the bivector e_{12}. The analogy goes further and we make the unitary group $U(1)$ to correspond to a Clifford group, which will be called the **spinor group**, defined just like $U(1)$, namely $Spin(2) = \{s \in \mathfrak{Cl}_2{}^+ \mid s\bar{s} = 1\}$. In this situation, $\bar{s}$ is called the *Clifford conjugate* of s and, if $s = s_0 + s_1 e_1 + s_2 e_2 + s_{12} e_{12}$, then $\bar{s} = s_0 - s_1 e_1 - s_2 e_2 - s_{12} e_{12}$. Moreover, it can be shown that $SO(2) \approx Spin(2)$.

Note that two opposed elements in $Spin(2)$ describe the same rotation in $SO(2)$. For this reason, it is said that *Spin(2) is a two-fold cover of SO(2)*. (To get an intuitive approach of two-fold covers, think, for example, of your watch. The same hour - i.e. position of the hands - stands for two opposite positions of the Sun - A.M. and P.M.). Mathematically, this is written as $Spin(2)/\{\pm 1\} \approx SO(2)$.

4.5.3 Pinor Group, Pin(2)

Pinors are introduced in a very similar way to the spinors, but they contain elements from the entire algebra, not just its even part. Therefore, we have:

Definition 4.5.1 *The* **pinor group** *in the Clifford algebra* $\mathfrak{Cl}_2$ *is defined as the set:*

$$Pin(2) = \{u \in \mathfrak{Cl}_2 \mid \bar{u}u = 1\},$$

where if

$$u = u_0 + u_1 e_1 + u_2 e_2 + u_{12} e_{12},$$

its reverse, denoted by $\bar{u}$ is

$$\bar{u} = u_0 + u_1 e_1 + u_2 e_2 - u_{12} e_{12}.$$

It can be shown that if we take the *orthogonal* group, $O(2) = \{R \in M_2(\mathbb{R}) \mid R^T R = I_2\}$, then the pinor group is a two-fold cover of it and it is isomorphic to it, i.e.

$$Pin(2)/\{\pm 1\} \approx O(2) \text{ and } Pin(2) \approx O(2).$$

Chapter 5

Exterior Algebras

In this chapter we will present formally the notion of an algebra endowed with exterior (vector) products. We will study real algebras and see how these constructs are similar to Clifford algebras. By doing this, we will follow somehow the historical development of the subject, since it was first the use of exterior algebras by H. Grassmann and then the further refinement into Clifford algebras by W. Clifford.

5.1 Bivectors Revisited

In this section we want to extract the basic properties of bivectors, in order to define them properly and to give further constructions based on them.

The key property of a bivector is that the square of its magnitude is negative 1. Remember, that when we first introduced bivectors (see **Section 4.1**), we said that $e_1e_2 = -e_2e_1$, or, as we denoted earlier, $e_{12} = -e_{21}$. This may give an idea of how bivectors are formed. If you look at the properties of the exterior product (more precisely, the skew-symmetry, see **Definition 2.7.3**), it is natural to consider that the "birth" of the bivector e_{ij} comes from the exterior (wedge) product:

$$e_{ij} = e_i \wedge e_j.$$

Note that, also due to **Definition 2.7.3**, respectively, to the alternating nature of the wedge product, we will never have a bivector e_{ii}, because

$$e_{ii} = e_i \wedge e_i = 0.$$

If we take the canonical basis of $\mathbb{R}^3$, namely $\{e_1,\ e_2,\ e_3\}$, we can obtain the following bivectors:

$$\begin{aligned} e_{12} &= e_1 \wedge e_2 \\ e_{13} &= e_1 \wedge e_3 \\ e_{23} &= e_2 \wedge e_3 \end{aligned}$$

Due to the properties of the wedge product (skey-symmetry and alternating), we see that this actually is the whole set of bivectors that can be obtained from the canonical base of $\mathbb{R}^3$.

Since $\{e_1,\ e_2,\ e_3\}$ span the space $\mathbb{R}^3$, it is somehow natural to think that the bivectors obtained above span another space. Indeed, they span the "exterior" space $\bigwedge^2 \mathbb{R}^3$. This is a vector space (over the real field, for example). But since we also have a notion of multiplication of (bi)vectors, it becomes an algebra, which is called the **exterior algebra**. The superscript 2 in denoting the algebra comes from the number of terms considered in the exterior products when constructing the space (or, if you like, the "length" of the index that the base-bivectors have).

Note also that $\bigwedge^1 \mathbb{R}^3 = \mathbb{R}^3$, since there are no bivectors in it and that $\bigwedge^3 \mathbb{R}^3 = sp(\{e_{123}\})$, because any other combination can be obtained from it, using the properties of the wedge product. (E.g. $e_{132} = e_1 \wedge e_3 \wedge e_2$, and since the wedge product is associative and skew-symmetric, we can write successively: $e_1 \wedge (-e_2 \wedge e_3) = -e_1 \wedge e_2 \wedge e_3 = -e_{123}$.) There is also the convention that $\bigwedge^0 \mathbb{R}^3 = \mathbb{R}$.

Therefore, any element in the algebra can be written as a linear combination of the base-bivectors:

$$B = B_{11}e_{11} + B_{12}e_{12} + B_{23}e_{23}.$$

From a geometrical point of view, the wedge (exterior) product is like the cross (vector) product and thus it will produce a bivector that is perpendicular on the plane generated by the factors. We will further show how the inner product and the norm of bivectors is defined.

In order to get bivectors and to keep a connection with the real space $\mathbb{R}^3$, we start with $x_1,\ x_2,\ y_1,\ y_2 \in \mathbb{R}^3$. In order to get bivectors, we make the wedge products $x_1 \wedge x_2$ and $y_1 \wedge y_2$, for example. Now, we define the scalar product of these bivectors as:

$$< x_1 \wedge x_2, y_1 \wedge y_2 >= \begin{vmatrix} x_1 \cdot y_1 & x_1 \cdot y_2 \\ x_2 \cdot y_1 & x_2 \cdot y_2 \end{vmatrix},$$

where the dot product is the usual scalar product on $\mathbb{R}^3$. Note that, in particular, if $x_1 = y_1 = x$ and $x_2 = y_2 = y$, then $< x \wedge y, x \wedge y >= |x|^2 |y|^2 - (x \cdot y)$, where we have used the norm and the inner Euclidean product.

The norm (or area, as it is called) of the bivector B decomposed above, will be:

$$|B| = \sqrt{< B, B >} = \sqrt{B_{12}^2 + B_{13}^2 + B_{23}^2}.$$

5.2 The Oriented Volume Element

The geometrical interpretation of the triple wedge product $a \wedge b \wedge c$ is the oriented volume of the parallelipiped with edges $a,\ b,\ c$, as one may find when making the connection with the triple cross product $a \times b \times c$ in $\mathbb{R}^3$.

As we marked above, the triple wedge product is a member of the (exterior) vector space $\bigwedge^3 \mathbb{R}^3$, which is one-dimensional. Thus any member of it can be uniquely written as $V = ve_{123}$, whose norm will be $|v|$. We can extend the scalar product as before:

$$< x_1 \wedge x_2 \wedge x_3, y_1 \wedge y_2 \wedge y_3 >= \begin{vmatrix} x_1 \cdot y_1 & x_1 \cdot y_2 & x_1 \cdot y_3 \\ x_2 \cdot y_1 & x_2 \cdot y_2 & x_2 \cdot y_3 \\ x_3 \cdot y_1 & x_3 \cdot y_2 & x_3 \cdot y_3 \end{vmatrix}.$$

Now, $\| V \| = \sqrt{< V, V >}$.

We can make further connections with the usual Euclidean space, seeing the wedge product just like a cross product. Thus, the next expression may be familiar:

$$a \wedge b = \begin{vmatrix} e_{12} & e_{13} & e_{23} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Here, we have taken $a, b \in \mathbb{R}^3$ two arbitrary vectors and $a_{ij}, b_{ij},\ i, j = 1, 2, 3$ their corresponding coordinates in the canonical basis of $\mathbb{R}^3$.

5.3 The Exterior Algebra and the Clifford Algebra

We will now sum up the (exterior) alegbras that we presented so far and see how these are connected with the Clifford algebras. Remember that

$\bigwedge^0 \mathbb{R}^3 = \mathbb{R}$ and $\bigwedge^1 \mathbb{R}^3 = \mathbb{R}^3$. Thus, we have the following decomposition:

$$\bigwedge \mathbb{R}^3 = \mathbb{R} \oplus \mathbb{R}^3 \oplus \bigwedge^2 \mathbb{R}^3 \oplus \bigwedge^3 \mathbb{R}^3.$$

Notation	Contains	Basis
$\mathbb{R}$	scalars	1
$\mathbb{R}^3$	vectors	$e_1,\ e_2,\ e_3$
$\bigwedge^2 \mathbb{R}^3$	bivectors	$e_{12},\ e_{13},\ e_{23}$
$\bigwedge^3 \mathbb{R}^3$	volume elements	e_{123}

The algebra $\bigwedge \mathbb{R}^3$ is called the **exterior algebra**, because it is generated by exterior products. Note that from the table above, the direct summands that are involved in the decomposition of this algebra have dimensions 1, 3, 3 and 1, respectively and thus the dimension of the exterior algebra is 8. This is the first hint that it may have something to do with the Clifford algebra $\mathfrak{Cl}_3$, due to **Remark 4.2.1**. To make a further step, we observe that the exterior algebra $\bigwedge \mathbb{R}^3$ is an *associative* algebra with unity that also satisfies

$$e_i \wedge e_j = -\delta_{ij} e_j \wedge e_i,$$

for all $i,\ j$, where δ_{ij} is the well-known Kronecker symbol and e_i are the elements of the canonical basis of $\mathbb{R}^3$. Moreover, for $a \in \bigwedge^i \mathbb{R}^3$ and $b \in \bigwedge^j \mathbb{R}^3$, we have that

$$a \wedge b \in \bigwedge^{i+j} \mathbb{R}^3.$$

Recall now that in the Clifford algebra $\mathfrak{Cl}_3$, for any vectors in the canonical base of $\mathbb{R}^3$, we have:

$$e_i e_j = -e_j e_i, \text{ for } i \neq j \text{ and}$$
$$e_i e_i = 1,$$

where we have denoted by juxtaposition the Clifford product.

Thus, just like in the case of the exterior algebra $\bigwedge \mathbb{R}^3$, the canonical basis of $\mathbb{R}^3$ generates a basis for the Clifford algebra $\mathfrak{Cl}_3$, and the table above will look just the same, but instead of the wedge product we will have the Clifford product, denoted by juxtaposition. Therefore,

$$\mathfrak{Cl}_3 = \mathbb{R} \oplus \mathbb{R}^3 \oplus \bigwedge^2 \mathbb{R}^3 \oplus \bigwedge^3 \mathbb{R}^3.$$

Since this algebra will contain scalars, vectors, bivectors and volume elements (just like the exterior algebra $\bigwedge \mathbb{R}^3$), we say that this decomposition introduces on the algebra a *multivector* structure. Therefore, for any $v \in \mathfrak{Cl}_3$, we have a unique decomposition (due to **Theorem 2.1.1**):

$$v = v_0 + v_1 + v_2 + v_3 + v_4, \text{ where } v_k \in \bigwedge^k \mathbb{R}^3,\ k = 0, \cdots, 4.$$

5.4 Even and Odd Parts

Just like the exterior and the $\mathfrak{Cl}_2$ Clifford algebra the 8-dimensional $\mathfrak{Cl}_3$ has its decomposition in even and odd parts. Thus, there is

$$\text{the even part } \mathbb{R} \oplus \bigwedge^2 \mathbb{R}^3$$
$$; \text{ the odd part } \mathbb{R}^3 \oplus \bigwedge^3 \mathbb{R}^3.$$

Because there are different products involved, the even subalgebra $(\bigwedge \mathbb{R}^3)^+$ of $\bigwedge \mathbb{R}^3$ is commutative, but the corresponding subalgebra of the Clifford algebra, ${\mathfrak{Cl}_3}^+$ is not commutative, even if it has the same direct summands.

Instead, it can be proven that ${\mathfrak{Cl}_3}^+$ is isomorphic to the algebra of quaternions, $\mathbb{H} \approx {\mathfrak{Cl}_3}^+$. Recall the isomorphism $\mathbb{H} \approx \mathbb{R} \oplus \mathbb{R}^3$. ${\mathfrak{Cl}_3}^+ \approx \mathbb{R} \oplus \bigwedge^2 \mathbb{R}^3$, but (by the simplest way of vector spaces with the same dimension that are isomorphic) $\bigwedge^2 \mathbb{R}^3 \approx \mathbb{R}^3$. Hence the conclusion. (For a more compelling technical description, see items [3] and [8] from the Bibliography.)

5.5 Gradings

We saw earlier that the elements of the exterior algebra $\bigwedge \mathbb{R}^3$ satisfy the equality:

$$a \wedge b \in \bigwedge^{i+j} \mathbb{R}^3, \text{ for } a \in \bigwedge^i \mathbb{R}^3,\ b \in \bigwedge^j \mathbb{R}^3.$$

According to the definition of grading (**Definition 3.2.1**), we have that the exterior algebra $\bigwedge \mathbb{R}^3$ is $\mathbb{Z}$-graded.

In what concerns the Clifford algebra $\mathfrak{Cl}_3$, we have the following inclusions:

$$\begin{aligned} {\mathfrak{Cl}_3}^+ \cdot {\mathfrak{Cl}_3}^+ &\subset {\mathfrak{Cl}_3}^+ \ ; \\ {\mathfrak{Cl}_3}^- \cdot {\mathfrak{Cl}_3}^+ &\subset {\mathfrak{Cl}_3}^- \ ; \\ {\mathfrak{Cl}_3}^+ \cdot {\mathfrak{Cl}_3}^- &\subset {\mathfrak{Cl}_3}^- \ ; \\ {\mathfrak{Cl}_3}^- \cdot {\mathfrak{Cl}_3}^- &\subset {\mathfrak{Cl}_3}^+ \ . \end{aligned}$$

Thus, if we denote again $\mathfrak{Cl}_3{}^- = (\mathfrak{Cl}_3)_1$ and $\mathfrak{Cl}_3{}^+ = (\mathfrak{Cl}_3)_0$, we get again a $\mathbb{Z}_2$-grading (or even-odd) for $\mathfrak{Cl}_3$.

Moreover, we can reconstruct the Clifford algebra $\mathfrak{Cl}_3$ from the exterior algebra $\bigwedge \mathbb{R}^3$ in a unique manner, thus exhibiting an isomorphism of the algebras. Recall first that both $\mathbb{R}$ and $\mathbb{R}^3$ have unique copies, both in the exterior algebra and the Clifford algebra. Also, recall that the Clifford product of two vectors is the sum of the inner product and the exterior product, namely:

$$ab = a \cdot b + a \wedge b, \ \forall a, b \in \mathfrak{Cl}_3.$$

We can reconstruct the wedge (exterior) product:

$$x \wedge y = \tfrac{1}{2}(xy - yx) \in \textstyle\bigwedge^2 \mathbb{R}^3, \ \forall x, y \in \mathbb{R}^3.$$

It follows tthat the part $\bigwedge^2 \mathbb{R}^3$ has a unique copy in $\mathfrak{Cl}_3$. The part $\bigwedge^3 \mathbb{R}^3$ can also be constructed using the Clifford product, like:

$$x \wedge y \wedge z = \tfrac{1}{6}(xyz + yzx + zxy - zyx - xzy - yxz) \in \textstyle\bigwedge^3 \mathbb{R}^3, \ \forall x, y, z \in \mathbb{R}^3.$$

Thus, we have made correspondences of all the direct summands that appear in the decomposition of the exterior algebras with the same parts appearing in the Clifford algebra. We say that we have established an isomorphism of algebras *by description*, because we described (i.e. showed) how the corresponding parts are connected. It is summarized in the following table:

Part	$\mathbb{R}^3$	$\to$	$\mathfrak{Cl}_3$
$\mathbb{R}$	α	$\to$	α
$\mathbb{R}^3$	x	$\to$	x
$\bigwedge^2 \mathbb{R}^3$	$x \wedge y$	$\to$	$\frac{1}{2}(xy - yx)$
$\bigwedge^3 \mathbb{R}^3$	$x \wedge y \wedge z$	$\to$	$\frac{1}{6}(xyz + yzx + zxy - zyx - xzy - yxz)$

Remark 5.5.1 There is another way of sending the part $\bigwedge^3 \mathbb{R}^3$ from $\bigwedge \mathbb{R}^3$ to $\mathfrak{Cl}_3$, by:

$$x \wedge y \wedge z = \tfrac{1}{2}(xyz - zyx) \in \textstyle\bigwedge^3 \mathbb{R}^3, \ \forall x, y, z \in \mathbb{R}^3,$$

which is a recursive construction, with the help of an intermediate step:

$$x \wedge B = \tfrac{1}{2}(xB + Bx) \in \textstyle\bigwedge^3 \mathbb{R}^3, \ \forall B \in \textstyle\bigwedge^2 \mathbb{R}^3, \ x \in \mathbb{R}^3.$$

Chapter 6

Pauli and Dirac Matrices

6.1 Pauli Matrices

We now pause the mathematical description of the Clifford algebras and introduce some quantum mechanics fundamentals, in order to see how the Clifford algebras can be applied to describe the spinors, via the Pauli and Dirac matrices. The presentation below will involve a quantum-mechanics briefing, since it is designed only to define the Dirac matrices, in order to emphasize their relation with the 16-dimensional Clifford algebra. For a more detailed description using proper theoretical physics tools, see the Appendix.

Recall that the time-dependent Schrodinger equation (TDSE):

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi$$

where $H = -\dfrac{\hbar^2}{2m}\nabla^2 + V$ is the Hamiltonian operator whose physical meaning is of total energy.

The wave function is complex valued $\psi = \psi(\vec{r}, t) \in \mathbb{C}$ and the square of its norm integrated over a region in space gives the probability of finding the particle in that region.

The Stern-Gerlach experiment in 1922 showed that a beam of silver atoms splits in a magnetic field. The explanation for this was that silver atoms and electrons possess intrinsic angular momentum, which they called *spin*. The spin interacts with the magnetic field, causing the beam to go up or down, depending if the spin is parallel or oposite to the vertical magnetic field.

In an electromagnetic field characterised by $\vec{E}$ and $\vec{B}$, with scalar potential V and vector potential $\vec{A}$, the TDSE becomes (let $e > 0$ be the elementary charge, i.e. the absolute value of the electron charge):

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}[(-i\hbar\nabla - e\vec{A})^2]\psi - eV\psi.$$

And after 'squaring' the bracket, we get:

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}[-\hbar^2\nabla^2 + e^2A^2 + i\hbar e(\nabla\cdot\vec{A} + \vec{A}\cdot\nabla)]\psi - eV\psi.$$

Note that although this equation is a bit more 'advanced' than the original TDSE, in the sense that it includes the electromagnetic field, it does not yet involve the spin of the electron.

Let us now introduce the *general momentum* operator in quantum mechanics, with an expression similar to the one in analytical mechanics, i.e.

$$\vec{\pi} = \vec{p} - e\vec{A}, \text{ where } \vec{p} = -i\hbar\nabla.$$

in such a way that its components $\pi_k - p_k - eA_k$ satisfy

$$[\pi_1, \pi_2] = \pi_1\pi_2 - \pi_2\pi_1 = i\hbar eB_3 \text{ (permute 1,2,3 cyclically).}$$

In 1927, Wolfgang Pauli introduced another term to the TDSE, by first introducing what are now called *the Pauli spin matrices*:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Remark 6.1.1 As it is the case of Dirac matrices, the Pauli matrices are not unique. They are only supposed to satisfy some properties (some of which we will list) and equations, but it can be proved that there is more than one set of choices for them. We presented the above for simplicity.

Note that these matrices satisfy the following equations (the proofs involve elementary matrix algebra):

$$\sigma_1\sigma_2 = i\sigma_3, \text{ and}$$
$$\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}I_2.$$

Applying the above commutation and anticommutation relations, we get that:

$$\vec{\sigma} \cdot \vec{\pi} = \sum_{i=1}^{3} \sigma_i \pi_i \text{ , and}$$
$$(\vec{\sigma} \cdot \vec{\pi})^2 = \vec{\pi}^2 - \hbar e(\vec{\sigma} \cdot \vec{B}) \text{ , where}$$
$$\vec{\pi}^2 = \vec{p}^2 + e^2 A^2 - e(\vec{p} \cdot \vec{A} + \vec{A} \cdot \vec{p}).$$

Pauli replaced $\vec{\pi}^2$ with $(\vec{\sigma} \cdot \vec{\pi})^2$ in the 'more advanced' TDSE that we mentioned and got:

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m}[\vec{pi}^2 - \hbar e(\vec{\sigma} \cdot \vec{B})]\psi - eV\psi$$

In this form, the Schrodinger-Pauli equation above describes the spin of the electron in virtue of the term $\frac{\hbar e}{2m}(\vec{\sigma} \cdot \vec{B})$.

The matrix $\vec{\sigma} \cdot \vec{B}$ operates on matrices from $\mathcal{M}_{2,1}(\mathbb{C})$. The wave function $\psi(\vec{r}, t)$ sends space-time points to *Pauli spinors*, i.e.:

$$\psi(\vec{r}, t) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \in \mathcal{M}_{2,1}(\mathbb{C}).$$

6.2 Dirac Equation and Dirac Matrices

To sum up, the Schrodinger equation describes all atomic phenomena, except for those involving electromagnetism and relativity. The Schrodinger-Pauli equation takes care of electromagnetism by introducing the spin of the electron. We will now briefly present the progress in including relativity in the equation.

The relativistic phenomena are taken into consideration by starting from the following equation:

$$\frac{H^2}{c^2} - \vec{p}^2 = m^2 c^2.$$

By inserting the corresponing operators, H and $\vec{p}$ from the relations where they appeared in the previous sections, we get the so-called *Klein-Gordon* equation, which has the following form:

$$\hbar^2(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2)\psi = m^2 c^2 \psi.$$

In 1928, Dirac linearized the equation, by replacing it with the following first-order equation:

$$i\hbar(\gamma_0\frac{1}{c}\frac{\partial}{\partial t}+\gamma_1\frac{\partial}{\partial x_1}+\gamma_2\frac{\partial}{\partial x_2}+\gamma_3\frac{\partial}{\partial x_3})\psi = mc\psi.$$

The above *Dirac equation* implies the Klein-Gordon equation, provided that the objects γ_i satisfy the relations:

$$\gamma_0^2 = I,\ \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -I,$$
$$\gamma_i\gamma_j = -\gamma_j\gamma_i, \text{ for } i \neq j.$$

Dirac found a set of 4×4 matrices that satisfy the above relations, which are now called *Dirac matrices*:

$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\ \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$
$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},\ \gamma_3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

If we remember the Pauli-spin matrices, their relation with the Dirac matrices can be written as follows:

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},\ \gamma_j = \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix}.$$

Going back to the Dirac equation, if we put $x_0 = ct$, then it will have the condensed form

$$i\hbar\gamma_k\partial_k\psi = mc\psi, \text{ where } \partial_k = \frac{\partial}{\partial x_k}.$$

If we take into account an interaction with a magnetic field via the space-time potential $(A_0, A_1, A_2, A_3) = (\frac{1}{c}V, A_x, A_y, A_z)$, we employ the replacement $i\hbar\partial_k \to i\hbar\partial_k - eA_k$. With this, the Dirac equation takes its conventional form:

$$\gamma_k(i\hbar\partial_k - eA_k)\psi = mc\psi.$$

where the wave-function is now called a (Dirac) bi-spinor, having four components:

$$\psi(x) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \in \mathcal{M}_{4,1}(\mathbb{C}) \approx \mathbb{C}^4.$$

The Dirac equation takes into account relativistic phenomena and also the spin. It is used in describing spin $\frac{1}{2}$ particles (generally called *fermions*), like the electron.

Remark 6.2.1 As we mentioned in Remark 4.4.1, the above representation of Dirac matrices is not unique, nor is the relation which involves the Pauli matrices, because the 'definition' of the γ matrices is given by the requirements for linearization of the Klein-Gordon equation. Hence, any matrices from $\mathcal{M}_4(\mathbb{C})$ that satisfy those relations are appropriate. However, we chose this representation for simplicity.

6.3 Dirac Matrices and Clifford Algebra

We close this chapter with some remarks on the connection of the Dirac and Pauli matrices with the Clifford algebra $\mathfrak{Cl}_4$ over $\mathbb{R}^4$, which will be another step forward from our previous presentation of the Clifford algebra $\mathfrak{Cl}_3$ over $\mathbb{R}^3$.

First, we make the remark that the Dirac matrices $\gamma_i,\ i = 0, \cdots, 4$ span the entire space $\mathcal{M}_4(\mathbb{C})$, which is isomorphic to the vector space $\mathbb{C}^4$. Moreover, due to the properties and the definitions of Pauli matrices, we get that the set $\{\sigma_k, i\sigma_k\}$ spans the space of bivectors, isomorphic to $\bigwedge^2 \mathbb{R}^4$, since they have the same properties. A further generalization will allow us to say that the set $\{i\gamma_k\}$ spans the space of trivectors, or of volume elements, isomorphic to $\bigwedge^3 \mathbb{R}^4$. Making a step further, we say that the complex number i generates *quadrivectors*, so that $sp(\{i\}) \approx \bigwedge^4 \mathbb{R}^4$. Thus, we can summarize all these in the table below (j goes from 1 to 3 and k, from 0 to 3).

Remark 6.3.1 The objects used in the table satisfy

$$\begin{gathered} i = \gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_1\sigma_2\sigma_3, \\ \sigma_j = \gamma_j\gamma_0, \\ i^2 = -1. \end{gathered}$$

Dimension	Basis	$\approx$	Contains	Our notation
1	$\{1\}$	$\mathbb{R} = \bigwedge^0 \mathbb{R}^4$	scalars	$\mathcal{S}$
4	$\{\gamma_k\}$	$\mathbb{R}^4 = \bigwedge^1 \mathbb{R}^4$	vectors	$\mathcal{V}$
6	$\{\sigma_j, i\sigma_j\}$	$\bigwedge^2 \mathbb{R}^4$	bivectors	$\mathcal{B}$
4	$\{i\gamma_k\}$	$\bigwedge^3 \mathbb{R}^4$	trivectors (volume elements)	$\mathcal{T}$
1	$\{i\}$	$\bigwedge^4 \mathbb{R}^4$	quadrivectors	$\mathcal{Q}$

First, note that, in our notation, $\mathcal{S} \cap \mathcal{V} \cap \mathcal{B} \cap \mathcal{T} \cap \mathcal{Q} = \{0\}$, i.e. they share only the null element. Therefore, if we make the sum of the spaces, it will be direct, namely $\mathcal{S} \oplus \mathcal{V} \oplus \mathcal{B} \oplus \mathcal{T}$. Moreover, the resulting space will be of dimension 16. An educated guess will immediately say that we are dealing with a Clifford algebra, using the fact that $dim\mathfrak{Cl}_\mathfrak{n} = 2^n$ and $16 = 2^4$. Therefore:

$$\mathfrak{Cl}_4 \approx \mathcal{S} \oplus \mathcal{V} \oplus \mathcal{B} \oplus \mathcal{T} \oplus \mathcal{Q}.$$

With this important relation, we made the connection between the real Clifford algebra $\mathfrak{Cl}_4$ over $\mathbb{R}^4$, the exterior algebra $\bigwedge \mathbb{R}^4$ and the Dirac and Pauli matrices.

Remark 6.3.2 As in the case of the Clifford algebra $\mathfrak{Cl}_3$ over $\mathbb{R}^3$, the only difference between the Clifford algebra $\mathfrak{Cl}_4$ over $\mathbb{R}^4$ and the exterior algebra $\bigwedge \mathbb{R}^4$ is the product between its elements: the exterior algebra is endowed with the exterior (wedge) product, whereas the Clifford algebra possesses the Clifford product, made up of the scalar plus the exterior product.

In view of this connection, the Dirac and Schrodinger-Pauli equations can be treated using the Clifford algebra (or geometric algebra, as it is called) techniques that we presented so far. Moreover, geometrical interpretations of the 16-dimensional algebra can be made, along with a $\mathbb{Z}_2$ grading (even-odd) and other geometrical constructions.

Chapter 7

Summary

To sum up, in this work we presented the basic techniques that are specific to the algebraic and geometric treatment of exterior and Clifford algebras, along with their applications in quantum mechanics, or more precisely, in describing the spin of the electron and the relativistic behavior of the time dependent Schrodinger equation (TDSE, for short).

In Chapter 2, we presented an outline of the basic concepts related to vector spaces and introduced the scalar and the exterior product, to make the further generalisations more accessible.

In Chapter 3, we saw that the group of vectors in a vector space can be made a *ring*, and the structure we obtain is called an *algebra*. Just like vector spaces can be decomposed in direct summands, algebras can be *graded*. $\mathbb{Z}_2$-gradings are the most useful for the purpose of our work.

Chapter 4 puts an exclamation mark on a particular construction, which is called *bivector*, since it cannot be a scalar, nor a vector. The oddity of it is that the square of its magnitude (or the scalar product with itself) is negative 1. This object will allow us to construct the basic Clifford algebra $\mathfrak{Cl}_3$ over $\mathbb{R}^3$.

In Chapter 5, we presented some 'relatives' of the Clifford algebras, namely the *exterior* algebras. Our aim was to make the Clifford algebras more familiar, using many connections with the common Euclidean space $\mathbb{R}^3$. Moreover, we saw that both the Clifford algebra $\mathfrak{Cl}_3$ and the exterior algebra $\bigwedge \mathbb{R}^3$ exhibit a decomposition in even and odd parts, which allowed us to make their $\mathbb{Z}_2$-gradings.

Chapter 6 presents the main goal of this work, the application of the Clifford algebra in the Dirac equation. After a brief recollection of the TDSE, we presented the Pauli correction that allows the *spin* of the electron to be taken into account and then the Dirac approach, who also took into consideration relativistic aspects. For his celebrated equation, Dirac constructed some peculiar matrices $\gamma_0, \gamma_1, \gamma_2, \gamma_3$, which we examined afterwards and saw that along with the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$, provide our sought-after connection with the Clifford algebra, that is the 16-dimensional $\mathfrak{Cl}_4$ algebra over $\mathbb{R}^4$.

All in all, we placed the Dirac equation in the framework of Clifford algebras and viceversa, since $\mathfrak{Cl}_4$ is the context where the equation 'takes place', but also the Dirac matrices are the key ingredient in forming a base for this algebra. We also presented some basic tools for manipulating these structures along with their physical importance, in order to facilitate algebraic, geometric and theoretical-physics developments and improvements.

Appendix A

A.1 From Schrodinger to Pauli and Dirac

In this section we will present a more detailed path to the Dirac equation and the Dirac matrices, using tools from theoretical physics or more precisely, from the field of relativistic quantum mechanics.

We start with the Schrodinger equation for the free particle, namely:

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi.$$

From a relativistic standpoint, the relation between the total energy and the momentum is:

$$E^2 = m^2c^4 + c^2\vec{p}^2.$$

We recall the coordinate-representation expression for the total energy and the momentum, namely:

$$E = i\hbar \frac{\partial \psi}{\partial t} \text{ and } \vec{p} = -i\hbar \nabla.$$

Now we move the above description in the space-time representation, a 4-dimensional Minkowski space. In this case, the momentum operator will have four components, out of which the spatial ones are the same as in the 3-dimensional case. The 'temporal' component will be $\frac{1}{c}E$, thus, the four-dimensional momentum, as a quadrivector will be:

$$\vec{p} = (\frac{i\hbar}{c}\frac{\partial}{\partial t}, -i\hbar\partial^j), \text{ j=1,2,3.}$$

If we introduce this four-operator along with the energy in the relation between the total energy and momentum (and apply the operators for an arbitrary ψ), we get the so-called *Klein-Gordon equation*:

$$-\hbar\frac{\partial^2\psi}{\partial t^2} = (m^2c^4 - \hbar^2c^2\nabla^2)\psi.$$

One of the problems of this model is that it allows both positive and negative values of the energy, for the same $\vec{p}$, namely $E = \pm\sqrt{m^2c^4 + c^2\vec{p}^2}$, which gives that the particle can have its energy in $E = (-\infty, -mc^2] \cup [mc^2, \infty)$. This creates a forbidden barrier of width $2mc^2$.

The fact that the values of the total energy are not bounded below means that one can extract from the system as much energy as one desires, which is not true. Moreover, in between the two intervals, namely in $(-mc^2, mc^2)$, any particle (and hence, atom) would be unstable and allowed to pass to the negative interval, again, by emitting as much energy as it desires.

To solve the problems of the model, the English physicist P.A.M. Dirac postulated that the correct equation that is to be found must satisfy some conditions.

First of all, it must involve temporal and spatial derivatives in a symmetric manner, as if there were no differences in the entities. Therefore, the resulting equation will be a partial differential equation of order 1.

Secondly, the equation must be linear, such that it obeys the superposition principle.

And last, but not least, its solutions must have the same nature (i.e. behavior) as those of the Klein-Gordon equation, which was proven to be correct, in the sense of agreement with nature.

Suppose, for this matter, that the sought-for equation has the form:

$$i\hbar\frac{\partial\psi}{\partial t} = [-i\hbar c(\sum_j \alpha_j\partial_j) + \beta mc^2]\psi,\ (\text{j=1,2,3, } \partial_j = \frac{\partial}{\partial x_j}).$$

In the above equation, the Hamiltonian operator has the expression contained in the square brackets and $\alpha_{1,2,3},\ \beta$ are constants. Let's put further the α matrices in a vector form, with which the Hamiltonian becomes $H = -i\hbar c\vec{\alpha}\cdot\nabla + \beta mc^2$, where $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$.

At this point, we require both the α and β matrices to be Hermitian, because they are contained in the expression of the Hamiltonian, which is a self-adjoint operator. Thus, $\alpha_i^+ = \alpha_i,\ \beta^+ = \beta\ (i = 1, 2, 3)$.

If we 'square' the hypothetical equation such that it resembles to the Klein-Gordon equation, we get:

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = (-i\hbar c\vec{\alpha} \cdot \nabla + \beta mc^2)^2 \psi.$$

If one 'squares' the bracket, one observes that the equation above is equivalent to the Klein-Gordon equation provided that the α and β satisfy:

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}\mathbb{1},\ \text{i,j=1,2,3}$$
$$\{\alpha_i, \beta\} = 0,\ \text{i=1,2,3}$$
$$\beta^2 = \mathbb{1}$$

We only required the α and β to be constants, i.e. scalars, but to place the equation in the four-dimensional spacetime, we need them to be square matrices. Note that the first property of the alpha matrices is equivalent to $\alpha_i\alpha_j = -\alpha_j\alpha_i$, for $i \neq j$, which, in turn, after applying the determinant function leads to $det(\alpha_i\alpha_j) = det(-\alpha_j\alpha_i) = (-1)^n det(\alpha_i\alpha_j)$, where $n \times n$ is the dimension of the matrices. Therefore, the α matrices, as well as the β matrices, must have the dimension an even number, i.e. n=2,4,6,.... Since they must be linearly independent, for the case n=2, we get the Pauli σ matrices. For n=4, we will see that the Dirac matrices which we will introduce, formally help forming 16 independent matrices. These matrices span the Clifford algebra $\mathfrak{Cl}_4$, as we saw in Section 6.3.

At this point, we will follow Dirac who introduced the γ matrices, using the α and β. Therefore, let $\gamma_0 = \beta$ and $\gamma_i = \beta \cdot \alpha_i$, i=1,2,3, which obey:

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}\mathbb{1},\ \text{i,j=1,2,3}$$
$$\gamma_0^+ = \gamma_0,\ \gamma_j^+ = -\gamma_j \Leftrightarrow \gamma_i^+ = \gamma_0\gamma_i\gamma_0,\ \text{j=1,2,3, i=0,1,2,3.}$$

Accordingly, the Klein-Gordon equation can now be put in Dirac's form, hence the name of Dirac equation:

$$i\gamma_0\partial_0\psi = (-\sum_{j=1}^{3} \gamma_k\partial_k + \frac{mc}{\hbar}\mathbb{1}).$$

In the following section, we will examine some properties of the Dirac γ matrices.

A.2 Properties of Dirac Matrices

The formal definition of the Dirac matrices requires them to satisfy the relations above, which are derived from the conditions for the more general α and β matrices. In this section, we will use the properties of the Dirac matrices in order to reconstruct the 16 independent matrices which span $\mathfrak{Cl}_4$. Thus, let $\Gamma_i \in \mathcal{M}_4(\mathbb{C}),\ i = 1, ..., 16$, with $\Gamma_1 = I_4$. Further, let

1. $\Gamma_2 = i\gamma_1,\ \Gamma_3 = i\gamma_2,\ \Gamma_4 = i\gamma_3,\ \Gamma_5 = \gamma_0,$

2. $\Gamma_6 = i\gamma_2\gamma_3,\ \Gamma_7 = i\gamma_3\gamma_1,\ \Gamma_8 = i\gamma_1\gamma_2,$

3. $\Gamma_9 = \gamma_0\gamma_1,\ \Gamma_{10} = \gamma_0\gamma_2,\ \Gamma_{11} = \gamma_0\gamma_3,$

4. $\Gamma_{12} = i\gamma_0\gamma_2\gamma_3,\ \Gamma_{13} = i\gamma_0\gamma_3\gamma_1,\ \Gamma_{14} = i\gamma_0\gamma_1\gamma_2,\ \Gamma_{15} = \gamma_1\gamma_2\gamma_3,$

5. $\Gamma_{16} = i\gamma_0\gamma_1\gamma_2\gamma_3.$

be the 16 matrices, which satisfy some properties that follow directly from the respective ones that the γs obey, namely:

- $\Gamma_i^2 = I_4$, for all i,

- $\forall\Gamma_{i,j}\ \exists\Gamma_k$ such that $\Gamma_i\Gamma_j = a\Gamma_k$, where $a \in \{\pm1, \pm i\}$,

- $\Gamma_i\Gamma_j = \pm\Gamma_j\Gamma_i$, for all i, j,

- $\forall\Gamma_i,\ i \neq 1,\ \exists\Gamma_j,\ j \neq 1$ such that $\Gamma_j\Gamma_i\Gamma_j = -\Gamma_i$ or $\{\Gamma_i, \Gamma_j\} = 0$.

Some immediate and important consequences are that the trace of the Γ matrices is zero (one gets this after applying the trace function to equality d.) and that they are linearly independent. From the so-called 'Alternative Theorem' (which states that given an n-dimensional vector space and a system of n vectors from it, the system is a basis for the space *either* if it is independent *or* if it spans the space), we get that $\{\Gamma_i\}_{i=1,\dots,16}$ is a basis for the $2^4 = 16$-dimensional algebra (which is also a vector space) $\mathfrak{Cl}_4$.

The relation between the Dirac and the Pauli matrices that we shown in Chapter 6 is straightforward, once one chooses a representation for the matrices. However, as we mentioned, the representation need not be unique and it is not. Nevertheless, the invariance of the Dirac equation holds irrespective of the representation that one chooses for the matrices. This is the result of Pauli's theorem, which states that given two sets of representations for the Dirac matrices (let them be $\{\gamma_i\}$ and $\{\gamma_i'\}$, i=0,1,2,3), they differ by a unitary transformation induced by a unitary nonsingular matrix T, such that:

$$\gamma_i' = T\gamma_i T^{-1} \Leftrightarrow \gamma_i' T = T\gamma_i.$$

The proof involves taking T as $T = \sum_{i=1}^{16} \Gamma_i' A \Gamma_i$, where the Γ matrices are obtained from the set γ matrices. A is an arbitrary 4×4 matrix, such that it makes T nonsingular. Further, one makes steps involving mostly elementary matrix algebra, but the complete proof is not the scope of this work.

All in all, we detailed in this Appendix the results that were used in Chapter 6 of this work, presenting to some extent the path which lead to them, altogether giving a theoretical physics flavor to the facts in Chapter 6.

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