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Metode omologice în studiul algebrelor şi coalgebrelor

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Homological Methods in the Study of Algebras and Coalgebras

– Ph.D. Thesis –

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INTRODUCERE

Principalul subiect al acestei teze este studiul omologic al algebrelor și coalgebrelor necomutative care sînt relevante în (nelimitîndu-se, însă la) geometria algebrică necomutativă. Punctul de plecare îl constituie studiul algebrelor Koszul, folosind metodele dezvoltate în articolul [JPS], anume via *perechi Koszul*. Mai mult, pentru generalitate, în cea mai mare parte a lucrării, vom studia R-inele (unde R este un inel semisimplu), în locul $\mathbb{k}$-algebrelor (cu $\mathbb{k}$ corp comutativ).

Primul tip de probleme căruia ne adresăm este dualizarea rezultatelor cunoscute privitoare la R-inele Koszul și deci să definim și studiem primele proprietăți ale R-*coinelelor Koszul*. După ce au fost introduse de S. Priddy in 1970 ([Pr]), apoi generalizate în context necomutativ de către A. Beilinson, V. Ginzburg și W. Soergel ([BGS]), abordarea din [JPS] folosind perechi Koszul s-a dovedit a fi în consonanță cu celelalte. Mai precis, într-o pereche Koszul, prima componentă este o algebră Koszul (R-inel, mai general), conform teoremei 2.13 din articolul citat. Acesta a fost punctul de plecare al cercetării prezente, a găsi proprietăți pentru cea de-a doua componentă a unei perechi Koszul, precum și formularea unor caracterizări echivalente pentru aceasta. După așteptări, am demonstrat că este vorba despre ceea ce am numit un R-*coinel Koszul*, ale cărui proprietăți am demonstrat că se pot obține prin dualitate categorială din cele ale R-inelelor Koszul.

Așadar, din punct de vedere definitoriu și descriptiv, situația a fost completată, în sensul că o pereche Koszul este alcătuită dintr-un inel și un coinel Koszul, legate printr-un izomorfism de R-bimodule și care au unele proprietăți interrelaționate prin dualitate.

Următorul punct al cercetării noastre a fost găsirea unor aplicații și exemple pentru asemenea structuri. Primul caz pe care l-am studiat a fost acela al R-*(co)inelelor local finite*. Acestea sînt (co)algebre graduate în categoria R-bimodululelor, cu proprietatea că fiecare componentă omogenă este finit dimensională. Utilizînd noțiunea de dualitate liniară graduată (la stînga/dreapta), am demonstrat că un R-inel local finit este Koszul dacă și numai dacă dualul său liniar graduat (la stînga/dreapta) este un R^{op}-coinel Koszul. O primă aplicație a acestor rezultate este în categoria algebrelor de incidență pentru poseturi finite și graduate. Astfel, înzestrînd algebra de incidență a unui asemenea poset cu o structură de R-inel, am găsit un algoritm constructiv pentru a obține un *poset Koszul* (i.e. un poset a cărui algebră de incidență să fie Koszul), pornind de la unul cu aceeași proprietate.

În încercarea de a da cît mai multe caracterizări echivalente pentru R-(co)inele Koszul, am studiat mai departe inelul Ext asociat unui inel Koszul. Am demonstrat existența unui izomorfism care leagă caracterizările cunoscute cu una folosind acest inel Ext. Lucrînd cu acest inel are avantajul simplității și posibilității de calcul explicit în unele situații, dintre care am menționat pe acelea al unor inele monoidale asociate submonoizilor lui $\mathbb{Z}^n$. Așadar, pentru a ilustra aceasta, am construit un exemplu explicit și am obținut o prezentare cu generatori și relații pentru inelul Ext asociat unui inel monoidal concret.

În final, un alt tip de aplicații pe care l-am studiat sînt legate de (co)omologia Hochschild a inelelor Koszul. Astfel, urmînd abordarea din [JPS], pentru a calcula (co)omologia Hochschild a unei algebre Koszul utilizînd o variație a complexului standard, am utilizat metodele respective în cazul unui produs tensorial răsucit de R-inele. Este deja cunoscut că, dacă (A, C) și (B, D) sînt perechi Koszul, atunci $(A \otimes_\sigma C, B \otimes_\tau D)$ este de asemenea o pereche Koszul, pentru anumite aplicații de răsucire $\sigma : B \otimes A \to A \otimes B$ de inele Koszul și $\tau : D \otimes B \to B \otimes D$ aplicații de răsucire (engl. *entwining*) de coinele Koszul. Am lucrat într-un conext mai simplu, undeva „între" cazul perechii inițiale (A, C) și perechea obținută din produs răsucit $(A \otimes_\sigma C, B \otimes_\tau D)$, luînd un R-inel arbitrar B și studiind proprietățile perechii $(A \otimes_\sigma B, C \otimes B)$, în ipoteza că perechea (A, C) este Koszul. Pentru aceasta, putem demonstra că perechea este aproape Koszul fără condiții suplimentare în ipoteză și mai mult, acest rezultat poate fi folosit pentru a pune complexul Hochschild standard pentru $A \otimes_\sigma B$ într-o formă particulară, legată de cazul anterior al perechii răsucite.

Să mai adăugăm că studiul algebrelor Koszul se poate face din numeroase puncte de vedere, revelînd multiple aplicații și exemple. Un caz, în strînsă legătură cu algebra comutativă și combinatorica, articole precum [CTV], [Fr], [Ci] sau capitolul [BjG] ori cartea [BH] oferă numeroase aplicații și rezultate teoretice de interes în studiul mulțimilor parțial ordonate sau al algebrelor de polinoame. De cealaltă parte, trimiteri semnificative și foarte recente către geometria necomutativă se fac în [RRZ], unde algebrele Koszul sînt cazuri particulare ale unor algebre (strîmb) Calabi-Yau, care, la rîndul lor, au fost introduse spre a generaliza în context necomutativ varietățile complexe Calabi-Yau. De asemenea generalizări, însă în direcția modificării condițiilor definitorii pentru algebrele Koszul au fost începute în mod semnificativ de Roland Berger, în articole precum [Bnq], [Bnq2], [BCo], iar apoi, în colaborare, în [BeG], [BDW] și, respectiv, [BM]. Alte generalizări ale algebrelor Koszul, care permit ca relațiile de algebră să poată fi date de elemente omogene de grad 1 și 2 au fost introduse în [CS] și [HR].

O abordare foarte modernă, folosind *teoria operazilor* a fost propusă în [GKa] și prezentată pe larg în cartea [LV]. Aplicații ale algebrelor Koszul pentru studiul grupurilor cuantice au apărut încă din 1987, în articole precum [Ma].

În plus, o direcție de studiu care deschide, la rîndul său, numeroase alte căi de abordare este *teoria deformărilor de algebre*. Începută prin seria de patru articole ale lui Murray Gerstenhaber ([G1], [G2], [G3], [G4]), cuprinse apoi în [GS], precum și prin contribuțiile lui Thomas Fox din [F], teoria și-a găsit aplicații în cazul algebrelor Koszul, prin articole precum [BG]. În acest articol, în urma reformulării teoremei Poincaré-Birkhoff-Witt folosind teoria deformării, se constată că dacă se pornește cu algebre Koszul, rezultatul are o formă particulară de interes. Subiectul a fost continuat în articole precum [FV].

În fine, dar nu cel mai din urmă, algebrele Koszul au fost studiate și prin intermediul teoriei de (co)omologie Hochschild, atît într-un context mai general, precum acela al N-complexelor, cît și pentru algebre particulare, precum algebre triunghiulare de matrice. În acest sens, referințe relevante includ (dar nu se limitează la): [GG], [Iov], [KW], [MP], [MM], [SW].

Teza este structurată după cum urmează. În primul capitol, prezentăm principalele preliminarii, definind structurile care se vor folosi pe întreg parcursul lucrării, precum și formulînd și demonstrînd cîteva rezultate de bază. Aceasta va include o familiarizare cu perechile (aproape) Koszul, unealta prin care majoritatea rezultatelor au fost obținute. Vom include, de asemenea, și o prezentare a unor structuri de bază și a proprietăților lor, precum R-(co)inelele graduate, condiția pentru graduarea tare, (co)inele bigraduate, (bi)comodule (tare) graduate, dualul liniar graduat al unui R-(co)inel local finit, complexul normalizat de (co)lanțuri și unele exemple. Mai mult, aplicațiile de răsucire și împletire sînt definite, deoarece vor fi folosite pentru a da exemple particulare, atît pentru (co)inele Koszul, cît și în (co)omologia Hochschild.

În capitolul al doilea, formulăm și demonstrăm șapte caracterizări echivalente pentru R-inele Koszul, ca rezultate preliminare pentru introducerea R-coinelelor Koszul într-o secțiune ulterioară. Așadar, mare parte a rezultatelor și demonstrațiilor privitoare la aceste ultime structuri vor fi formulate prin intermediul dualității relative la primele. Mai mult, vom extinde lista caracterizărilor, implicînd și inelul Ext al unui inel Koszul, precum și unele legături între inele și coinele Koszul, prin intermediul dualității liniare graduate, în cazul local finit.

Al treilea capitol al lucrării prezintă exemple și aplicații pentru rezultatele conținute mai sus, precum și o completare a prezentării cu studiul (co)omologiei Hochschild pentru inele Koszul. Ca o primă aplicație a rezultatelor privitoare la dualitatea liniară graduată, studiem poseturile finite și graduate, pe care le vom trata cu uneltele perechilor (aproape) Koszul dezvoltate pînă aici. Mai mult, vom formula un algoritm constructiv pentru a obține un poset care are (co)inelul de incidență Koszul (caz în care posetul însuși se va numi Koszul). Vom pune accent pe situațiile în care algoritmul funcționează și este relevant, dar îi vom evidenția și unele limitări și excepții. În final, privitor la inelul Ext al unui inel Koszul, îi vom exemplifica unele utilizări în cazul unui inel monoidal concret asociat unui submonoid al lui $\mathbb{Z}^n$. Secțiunea ultimă a capitolul al treilea deschide studiul în direcția (co)omologiei Hochschild pentru inele Koszul. Amintind unele rezultate din [JPS], care descriu complexul standard în cazul Koszul, studiem situația produselor tensoriale (eventual răsucite) de inele Koszul, în scopul înțelegerii structurii de inel pe coomologia Hochschild.

Rezultatele originale pe care această teză se bazează sînt concentrate în articolele:

(1) [M] – Utilizate în §2.4, conțin caracterizări ale inelului Ext asociat unui R-inel Koszul. Exemplele oferite sînt în categoria poseturilor Koszul și inelelor monoidale întregi;

(2) [MS1] – Precum prezentăm în §2.2, lucrarea introduce noțiunea de coinele Koszul, ca o dualizare naturală a inelelor Koszul. Mai mult, lucrarea conține caracterizări echivalente și exemple în cazul R-inelelor local finite;

(3) [MS2] – După formularea și demonstrarea caracterizărilor echivalente pentru inele Koszul, lucrarea prezintă metoda algoritmică de obținere a exemplelor de inele (și coinele) Koszul, în categoria inelelor de incidență pentru poseturi finite și graduate. Rezultatele sînt conținute aici în §3.1.

Nu pot trece la conținutul principal al tezei fără a formula mulțumirile care se cuvin. Mai întîi, recunoștință sinceră și profundă către Prof. Dragoș Ștefan, care m-a îndrumat cu atenție încă din timpul studiilor masterale, eu venind de la Facultatea de Fizică și care, de asemenea, mi-a coordonat și teza masterală. Aceeași recunoștință se cuvine și Prof. Sorin Dăscălescu, care m-a ajutat să acced și să mă integrez în Facultatea de Matematică, implicîndu-mă în proiecte de cercetare și oferindu-mi de fiecare dată sfaturi neprețuite. De asemenea, Domnii Profesori Victor Alexandru, Marian Aprodu, Liviu Ornea, Victor Ţigoiu și Victor Vuletescu, precum și mulți alți profesori și personal al facultății au contribuit adesea la devenirea mea, cu discuții, sfaturi și idei dintre cele mai utile.

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INTRODUCTION

The main scope of this thesis is to study homological properties of noncommutative algebras and coalgebras that are relevant in (but not limited to) Noncommutative Algebraic Geometry. The starting point is the study of Koszul algebras, using the methods developed in the article [JPS], namely via *Koszul pairs*. Moreover, for generality, in most of the work, we study R-rings (where R is a semisimple ring), instead of $\Bbbk$-algebras (for $\Bbbk$ a field).

The first type of problem that we addressed is to dualise the results that are known regarding Koszul R-rings and thus to define and research the first properties of *Koszul* R*-corings*. After the introduction of Koszul algebras by S. Priddy in 1970 ([Pr]) and the generalisation in a noncommutative framework by A. Beilinson, V. Ginzburg and W. Soergel ([BGS]), the approach in [JPS] that uses Koszul pairs was proved to be consistent with the other ones. That is, in a Koszul pair, the first component is a Koszul algebra (R-ring, more generally), by Theorem 2.13 in the cited article. This was the starting point for our present research, finding more properties for the second component of a Koszul pair and provide equivalent characterisations for it. As expected, it turned out to be what we called a *Koszul* R*-coring* whose properties we proved to be obtained by categorical duality from those of Koszul R-rings.

Therefore, from the defining and descriptive point of view, the picture was completed, in the sense that a Koszul pair consists of a Koszul ring and a Koszul coring, connected by an R-bimodule isomorphism and which shared many properties by duality.

The next point of our research was finding applications and examples for such structures. The first such case which we studied was of *locally finite* R*-(co)rings*. These are graded (co)algebras in the category of R-bimodules for which every homogeneous component is finitely dimensional. Using the notion of (left) graded duality, we proved that a locally finite R-ring is Koszul if and only if its (left) graded dual is a Koszul R^{opp}-coring. A first application was in the field of incidence algebras for finite graded posets. Thus, endowing the incidence algebra of such a poset with an R-ring structure, we have provided a constructive algorithm for obtaining a *Koszul poset* (i.e. a poset whose incidence algebra is Koszul) starting from one with the same property.

In the attempt of extending the equivalent characterisations for Koszul R-(co)rings, we have studied further the Ext ring associated to a Koszul ring. We have proved an isomorphism which intertwines the existent characterisations with one involving the Ext ring. Working with this former ring has the advantage of simplicity and effective computability in some cases, of which we mention that of monoid rings associated to submonoids of $\mathbb{Z}^n$. Thus, to illustrate this, we have computed an explicit example and obtained a presentation by generators and relations for the Ext ring of a particular concrete monoid ring.

Finally, another type of applications which we studied are related to Hochschild (co)homology for Koszul rings. Thus, following the approach developed in [JPS] to compute the Hochschild (co)homology of a Koszul algebra using a modified version of the standard complex, we employed these methods in the case of twisted tensor products of R-rings. It is already known that if (A, C) and (B, D) are Koszul pairs, then $(A \otimes_\sigma C, B \otimes_\tau D)$ is Koszul as well, for certain twisting map $\sigma : B \otimes A \to A \otimes B$ of Koszul rings and $\tau : D \otimes B \to B \otimes D$ an entwining map of Koszul corings. We worked in a simpler setting, somewhere "between" the initial pair (A, C) and the twisted tensor product pair $(A \otimes_\sigma C, B \otimes_\tau D)$, using an arbitrary R-ring B and study the properties of the pair $(A \otimes_\sigma B, C \otimes B)$, provided that the pair (A, C) is Koszul. For this, we can show that the pair is almost-Koszul without any further conditions and moreover, this result can be used to put the standard Hochschild complex for $A \otimes_\sigma B$ in a specific form, somehow related to the previous case above.

Let us add that the study of Koszul algebras can be performed from multiple points of view, thus revealing multiple applications and examples. One case, closely related to commutative algebra and combinatorics, articles such as [CTV], [Fr], [Ci] or the chapter [BjG] or the book [BH] offer many applications and theoretical results of interest in the study of partially ordered sets or polynomial algebras. On the other hand, significant and very recent directions towards noncommutative geometry can be found in [RRZ], where one finds Koszul algebras as particular cases of some (skew) Calabi-Yau algebras, which in turn were introduced to generalise in a noncommutative setup the complex Calabi-Yau manifolds. Other generalisations, but in the direction of modification of the defining conditions for Koszul algebras were started significantly by Roland Berger in articles such as [Bnq], [Bnq2], [BCo] and then, in collaboration, in [BeG], [BDW] and [BM], respectively. Other generalisations of Koszul algebras, which allow for the algebra relations to be given by homogeneous elements of degree 1 and 2 were introduced in [CS] and [HR].

A very modern approach, using *operad theory* was proposed in [GKa] and presented at length in the book [LV]. Applications of Koszul algebras for the study of quantum groups appeared since 1987, in articles such as [Ma].

Furthermore, a direction of study which opens, in itself, many other routes of approach is the *algebra deformation theory*. Started by the series of four articles of Murray Gerstenhaber ([G1], [G2], [G3], [G4]), then surveyed in [GS], as well as through the contributions of Thomas Fox in [F], the theory found applications in the case of Koszul algebras, by articles such as [BG]. In this article, following a reformulation of the Poincaré-Birkhoff-Witt theorem using deformation theory, it is remarked that when one starts with a Koszul algebra, the result has a particular form of interest. The subject was continued in articles such as [FV].

Last, but not least, Koszul algebras were studied also through Hochschild (co)homology theory, both in a general context, such as that of N-complexes, and also for particular algebras, such as triangular matrix algebras. In this sense, relevant references include (but are not limited to): [GG], [Iov], [KW], [MP], [MM], [SW].

The thesis is structured as follows. In the first chapter we provide the basic preliminaries, defining the structures which will be used throughout the work, as well as stating the basic results that are required. This includes a basic familiarisation with *(almost) Koszul pairs*, the tool by means of which most of the results are obtained. It also includes recalling more basic structures and properties, such as graded R-(co)rings, the condition for strong grading, bigraded (co)rings, (strongly) graded (bi)comodules, the graded linear dual of a locally finite R-(co)ring, the normalised (co)chain complex and some examples. Moreover, twisting, entwining and braiding maps are defined, since they will be used to provide the more specialised results later on, both for Koszul (co)rings and for Hochschild (co)homology.

In the second chapter, we state and prove seven equivalent characterisations for Koszul R-rings, as a preliminary work for the introduction of Koszul R-corings in a subsequent section. Hence, most of the results and proofs regarding the latter structures will be formulated by means of duality with respect to those for the former. Moreover, we will provide further extension of the list of characterisations, involving the Ext ring of a Koszul ring, as well as some connections between Koszul rings and corings by means of graded linear duality, in the case of local finiteness.

The third chapter of the thesis presents illustrative examples and applications for the results contained above, as well as enrich the presentation with the study of Hochschild (co)homology for Koszul rings. A first application of the results concerning the graded linear dual are in the direction of finite graded partially ordered sets (posets), which we will see that can be treated with the tools of (almost) Koszul pairs developed thus far. Moreover, we will provide a constructive algorithm for obtaining a poset which has a Koszul incidence (co)ring (case in which the poset will be called *Koszul*). We will emphasise the situations in which the algorithm works and is relevant, as well as some of its limitations and exceptions. Finally, concerning the Ext ring of a Koszul ring, we will exemplify its use with a concrete example in the category of monoid rings associated to submonoids of $\mathbb{Z}^n$, a finite direct sum of copies of the integers monoid. The very last section of the third chapter opens the study in the direction of the Hochschild (co)homology for Koszul rings. Recalling some results from [JPS] which describe what the standard complex becomes in the Koszul case, we then study the situation of (twisted) tensor products of (Koszul) rings, in the scope of a better understanding the ring structure on the Hochschild cohomology.

The original research papers on which the thesis is based are:

(1) [M] – Used in §2.4, it shows basic characterisations of the Ext ring associated to a Koszul R-ring. Examples are provided in the category of Koszul posets and integer monoid rings;

(2) [MS1] – As presented in §2.2, the paper introduces the notion of Koszul corings, as natural dualisations of Koszul rings. Furthermore, it provides several equivalent characterisations and examples in the case of locally finite R-rings;

(3) [MS2] – After formulating and proving equivalent characterisations of Koszul rings, the paper provides an algorithmic method of obtaining examples of Koszul rings (and corings), in the category of incidence rings for finite graded posets. The results are presented in §3.1.

I cannot pass to the main content of the thesis without mentioning due acknowledgements. First, my gratitude towards Prof. Dragoş Ştefan, who has carefully guided me ever since my master's years, coming from Physics and who also has coordinated my M.Sc. thesis. More of the same thankfulness goes to Prof. Sorin Dăscălescu, who has helped me come and integrate to the Faculty of Mathematics, involving me in research projects and offering invaluable advice. Furthermore, Professors Victor Alexandru, Marian Aprodu, Liviu Ornea, Victor Ţigoiu and Victor Vuletescu, as well as many other professors and staff of the Faculty have always contributed with careful insight, advice and fruitful discussions.

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CHAPTER 1

PRELIMINARIES

After their introduction by S. Priddy in [Pr], Koszul algebras proved to be a very useful tool in many fields of mathematics, such as representation theory, algebraic geometry, algebraic topology, quantum groups and combinatorics, commutative algebra. As such, their first uses was connected to polynomial algebras and quotients of these, then in the study of incidence algebras for posets. Under the influence of [BGS], many more opportunities opened in the field of noncommutative algebra and geometry. As such, noncommutative Koszul algebras proved to be useful in the study of noncommutative Calabi-Yau algebras, introduced by V. Ginzburg in [GCY]. Seen initially by analogy as a noncommutative counterpart of the analogous structures in complex geometry, Calabi-Yau algebras are intensively studied also for their purely algebraic structures. And in this respect, Koszul algebras make a basic introduction.

Let us recall the basic definition in [BGS], which provides the groundwork for the approach we took.

Definition 1.1 ([BGS], Definition 1.1.2)**:** A Koszul ring is a positively graded ring decomposed as $A = \oplus_{j \geq 0} A^j$ such that A^0 is semisimple and A^0 has a projective resolution in the category of graded left A-modules:

$$0 \leftarrow A^0 \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \dots,$$

such that P_i is generated by its degree i component, i.e. $P_i = AP_i^i$.

1.1 General Notions and Notations

1.1.1 (Bi)Graded (Co)Rings

While studying some cohomological properties of Koszul rings in [JPS], the authors were led to a new construction, that of *Koszul pairs*, which we now detail. Let R be a semisimple ring. Instead of Koszul rings or Koszul $\Bbbk$-algebras (with $\Bbbk$ a field), we will work with graded R-rings.

By "graded" we will always mean $\mathbb{N}$-*graded*, unless otherwise specified. Another terminology remark is that a (co)algebra will always be (co)associative and (co)unital, unless otherwise specified.

Recall that a graded R-ring is a graded algebra in the category of R-bimodules. That is, it has an algebra structure such that the multiplication is R-bilinear. Such an R-ring $A = \oplus_{n\in\mathbb{N}} A^n$ is called *connected* if $A^0 = R$. By duality, an R-coring is a coalgebra in the same tensor category of R-bimodules and a graded connected R-coring $C = \oplus_{n\in\mathbb{N}} C_n$ means that $C_0 = R$.

Unadorned tensor products will be considered over the base ring R. Also, to ease the notation, for any R-bimodule X, we will write $X^{(n)}$ and mean the n-fold tensor product over R of X with itself. Another piece of notation is that we will write Id_X for the identity map of the set X or, when there is no risk of confusion, we will denote it (abusively) by X itself.

For a graded, connected R-ring A, the multiplication map $\mu : A \otimes A \to A$ has homogeneous components $\mu^{p,q} : A^p \otimes A^q \to A^{p+q}$, which are R-bilinear, for any $p, q \in \mathbb{N}$. We call a graded connected R-ring A *strongly graded* if all such components are surjective. Equivalently, A is strongly graded if and only if $\mu^{p,1}$ is surjective, for all $p \in \mathbb{N}$.

Let $\pi_A^n : A \to A^n$ be the canonical projection. We will denote by A_+ the ideal $\oplus_{n>0} A^n$, which can be seen to coincide with the quotient A/A^0. Then, the multiplication of A induces an R-bimodule map $\mu_+ : A_+ \otimes A_+ \to A_+$.

As usual, one could also work with the opposite structure. If $(A, \mu, 1)$ is an R-ring, then we can consider the R^{op}-ring A^{op} whose multiplication is defined by the equation $\mu^{op}(a \otimes b) = \mu(b \otimes a)$, $\forall\, a, b \in A^{op} = A$.

One can dualise all of the above. Consequently, the comultiplication map Δ of a graded connected R-coring C has homogeneous components $\Delta_{p,q} : C_{p+q} \to C_p \otimes C_q$, for all non-negative integers p, q, which are R-bilinear maps. C is called *strongly graded* if all of these components are injective or, equivalently, if $\Delta_{p,1}$ is injective, for all non-negative integer p.

The subcoalgebra C_+ is defined as $C_+ = \oplus_{n>0} C_n$ which is further identified with C/C_0 and the comultiplication induces the corresponding R-bimodule map $\Delta_+ : C_+ \to C_+ \otimes C_+$.

One can further consider the opposite structure as well, defined in an entirely similar way to the dual case above.

It is useful to adapt Sweedler's notation for the image of an element through the comultiplication for the graded case. As such, for $c \in C_{p+q}$, we will write:

$$\Delta_{p,q}(c) = \sum c_{1,p} \otimes c_{2,q}, \tag{1.1}$$

which is to be understood as the first component of the tensor monomials being an element of C_p and the second belonging to C_q. Graded coassociativity now reads:

$$\sum c_{1,p+q_{1,p}} \otimes c_{1,p+q_{2,q}} \otimes c_{2,r} = \sum c_{1,p} \otimes c_{2,q+r_{1,q}} \otimes c_{2,q+r_{2,r}}, \ \forall p, q, r \in \mathbb{N},\ c \in C_{p+q+r}. \tag{1.2}$$

By definition of connected corings, the unit of R must be a group-like element, so $\Delta(1) = 1 \otimes 1$, see [JPS, §1]. Thus it makes sense to consider primitive elements of C, i.e. those $c \in C$ for which $\Delta(c) = c \otimes 1 + 1 \otimes c$. We will denote by PC the set of primitive elements of C and remark that $C_1 \subseteq PC$, the inclusion being strict, in general.

The dual notion of primitive elements is defined by J. P. May in [May], as follows:

Definition 1.2 ([May], Definition 2.5)**:** Keeping the context and the notation above, the graded R-bimodule of *indecomposable elements* of A is denoted by QA and defined by the exact sequence:

$$A_+ \otimes A_+ \xrightarrow{\mu_+} A_+ \to QA \to 0.$$

Note that there is a canonical morphism $A^1 \to QA$, which maps $a \in A^1$ to its equivalence class $a + A_+^2 \in QA$, which has a left inverse $QA \to A^1$, induced by the projection $A_+ \to A^1$.

Another structure of interest is that of the *shriek (co)ring*. We will define it in a general context, then explain how it appears in our work. Let V be any R-bimodule and $W \subseteq V \otimes V$ be a sub-bimodule. The R-ring $\langle V, W \rangle$ is defined as $T_R^a(V)/\langle W \rangle$, i.e. the quotient of the tensor algebra of the R-bimodule V by the ideal generated by W. The latter is a sum of homogeneous components denoted by $\langle W \rangle^n$, obtained as:

$$\langle W \rangle^n = \sum_{i=1}^{n} V^{(i-1)} \otimes W \otimes V^{(n-i-1)}.$$

Keeping the notations and the context, one can define by duality a graded R-coring $\{V, W\}$, whose homogeneous component of degree zero is $\{V, W\}_0 = R$, the next one is $\{V, W\}_1 = V$ and for all $n \geq 2$ define:

$$\{V, W\}_n = \bigcap_{p=1}^{n-1} V^{(p-1)} \otimes W \otimes V^{(n-p-1)}.$$

It is shown in the proof of Theorem 2.13 in [JPS] that the direct sum denoted by $\{V, W\} = \bigoplus_{n \in \mathbb{N}} \{V, W\}_n$ is a graded subcoring of the tensor coalgebra of the R-bimodule V, which is denoted by $T_R^c(V)$.

The particular cases we are interested in are defined in the following:

Definition 1.3: Let A be a graded connected R-ring and C be a graded connected R-coring. Keeping the notations above, the graded R-coring $A^! = \{A^1, \mathrm{Ker}\mu^{1,1}\}$ is called the *shriek coring* of A. By duality, the graded R-ring $C^! = \langle C_1, \mathrm{Im}\Delta_{1,1} \rangle$ is called the *shriek ring* of C.

The shriek structures are also graded and we will denote the homogeneous components of degree n by $A_n^!$ and $C_n^!$, respectively, for all n.

We will also need some notions regarding *bigraded corings*, which we state in a more general context as well. Start with the definition.

Definition 1.4: The R-coring C is called *bigraded* if it has a decomposition as a direct sum of R-bimodules $C = \bigoplus_{m,n \in \mathbb{N}} C_{n,m}$ such that its comultiplication induces a collection of maps of the form:

$$C_{n+n',m+m'} \xrightarrow{\Delta_{n,m}^{n',m'}} C_{n,m} \otimes C_{n',m'}.$$

For such a coring, coassociativity is represented as the commutativity of the diagram below, for all positive integers m, n, p, m', n', p':

$$\begin{array}{ccc} C_{n+m+p,n'+m'+p'} & \xrightarrow{\Delta^{p,p'}_{n+m,n'+m'}} & C_{n+m,n'+m'} \otimes C_{p,p'} \\ \Big\downarrow \Delta^{m+p,m'+p'}_{n,n'} & & \Big\downarrow \Delta^{m,m'}_{n,n'} \otimes \mathrm{Id}_{C_{p,p'}} \\ C_{n,n'} \otimes C_{m+p,m'+p'} & \xrightarrow[\mathrm{Id}_{C_{n,n'}} \otimes \Delta^{p,p'}_{m,m'}]{} & C_{n,n'} \otimes C_{m,m'} \otimes C_{p,p'} \end{array}$$

Also, by definition, the counit must vanish on all $C_{n,m}$ where either $n > 0$ or $m > 0$.

We introduce yet another notation. Let $\Delta(1) = \mathrm{Id}_{C_1}$ and for all $n \geq 2$ define $\Delta(n) : C_n \to C_1^{(n)}$ recursively as:

$$\Delta(n) = (\mathrm{Id}_{C_1} \otimes \Delta(n-1)) \circ \Delta_{1,n-1}.$$

From the very definition of the maps $\Delta(n)$, along with the coassociativity we obtain the equality:

$$\Delta(p+q) = (\Delta(p) \otimes \Delta(q)) \circ \Delta_{p,q}. \tag{1.3}$$

Therefore, using this notation, C being strongly graded amounts to $\Delta(n)$ being injective for all n.

Note that to any bigraded coring, one can associate a graded coring $\mathrm{gr}(C)$, which has the homogeneous component of degree n given by $\mathrm{gr}_n(C) = \oplus_m C_{n,m}$. Thence, $\mathrm{gr}(C) = \oplus_n \mathrm{gr}_n(C)$. Of exclusive interest for this thesis are the *connected bigraded corings*, that is bigraded corings for which $C_{0,0} = R$ and $C_{0,m} = 0$, whenever $m > 0$. When this is the case, it follows that $\mathrm{gr}(C)$ is also connected.

For any connected bigraded coring C, define:

$$C'_{n,m} = \begin{cases} C_{n,m}, & n = m \\ 0, & n \neq m. \end{cases}$$

With this at hand, $C' = \oplus_{n,m} C'_{n,m}$ is a connected bigraded coring and denote by $\mathrm{Diag}(C)$ the associated graded coring $\mathrm{gr}(C')$.

For all such corings C and C', there exist canonical R-bimodule isomorphisms denoted by $\pi_{n,m} : C_{n,m} \to C'_{n,m}$ which are identities on $C_{n,n}$ and zero in rest. Then the collection of all such maps, $\pi = \{\pi_{n,m}\}_{n,m}$ defines a morphism of bigraded corings and since $\mathrm{Diag}(C) = \mathrm{gr}(C')$, it follows that the map π induces a morphism from $\mathrm{gr}(C)$ to $\mathrm{Diag}(C)$, which is a map of graded corings. It is now clear that the kernel of $\mathrm{gr}(\pi)$ coincides with $\oplus_{n\neq m} C_{n,m}$.

We add the remark that the dual case of bigraded rings can also be considered (and in fact it is better known in the literature), but it will not be used in this paper, hence we omit its presentation.

1.1.2 Graded Comodules

Another type of algebraic structures which we will use throughout the thesis is that of *graded comodules* . Although the notion may be familiar in the ordinary (non-graded) case, to keep the writing as self contained as possible, we will present the basic definitions and properties in the graded case as well.

Following the definitions in [DNR, Chapter 2], for a coalgebra D over a field $\Bbbk$, a pair (M, ρ) is called a *right* D-*comodule* if M is a $\Bbbk$-vector space and a $\Bbbk$-linear map $\rho : M \to M \otimes D$ such that the following diagrams are commutative:

$$\begin{array}{ccc} M & \xrightarrow{\rho} & M \otimes D \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle Id_M \otimes \Delta_D} \\ M \otimes D & \xrightarrow{\rho \otimes Id_d} & M \otimes D \otimes D \end{array} \qquad \begin{array}{c} M \xrightarrow{\simeq} M \otimes \Bbbk \\ M \xrightarrow{\rho} M \otimes D \xrightarrow{Id_M \otimes \varepsilon} M \otimes \Bbbk \end{array}$$

where $\Delta_D : D \to D \otimes D$ is the comultiplication of the coalgebra D and $\varepsilon : D \to \Bbbk$ is its counit. Left D-comodules and D-bicomodules are defined similarly. We now pass to case of graded comodules over an R-coring.

Let C be an R-coring. We will denote the categories of left, right and bi-comodules over C by ${}^{C}\mathcal{M}$, $\mathcal{M}^{C}$ and ${}^{C}\mathcal{M}^{C}$, respectively. As in [JPS, §1.9], a pair (M, ρ^M) is called a *right* C-*comodule* if M is a right R-module and the map $\rho^M : M \to M \otimes C$ is a morphism of right R-modules such that similar diagrams to the above are commutative. Instead of repeating the diagrammatic illustration, we will state the conditions using the Sweedler notation for comodules. For the image of an element $m \in M$ we write $\rho^M(m) = \sum m_0 \otimes m_1$ and the right comodule condition means that the following equalities hold true (where $\varepsilon : C \to R$ is the counit of C):

$$\sum m_{0_0} \otimes m_{0_1} \otimes m_1 = \sum m_0 \otimes m_{1_1} \otimes m_{1_2}; \tag{1.4a}$$

$$\sum m_0 \varepsilon(m_0) = m. \tag{1.4b}$$

A morphism of right C-comodules is a right R-linear map which commutes with the comodule structure map ρ^M. An entirely similar construction is made for defining left C-comodules. The Sweedler notation for a left C-comodule (N, λ^N) is written as $\lambda^N(n) = \sum n_{-1} \otimes n_0$, $\forall n \in N$. Lastly, a C-*bicomodule* is a triple (M, λ^M, ρ^M) such that M is an R-bimodule, (M, λ^M) is a left C-comodule, (M, ρ^M) is a right C-comodule and the structures are compatible in the sense that for all $m \in M$, the following equality holds true:

$$\sum m_{-1} \otimes m_{0_0} \otimes m_{0_1} = \sum m_{0_{-1}} \otimes m_{0_0} \otimes m_1. \tag{1.5}$$

Note that by definition of C-comodules, the lateral maps ρ^M and λ^M are R-bimodule morphisms, by definition. Otherwise, the equality above does not make sense.

Finally, a morphism $f : M \to N$ of C-bicomodules is a map that is left and right C-colinear, that is it makes both of the squares in the diagram below commutative.

$$\begin{array}{ccccc} M \otimes C & \xleftarrow{\rho^M} & M & \xrightarrow{\lambda^M} & C \otimes M \\ \downarrow{\scriptstyle f \otimes Id_C} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Id_C \otimes f} \\ N \otimes C & \xleftarrow{\rho^N} & N & \xrightarrow{\lambda^N} & C \otimes N \end{array}$$

A result which is needed later on to prove that the Koszul complexes provide appropriate resolutions is the following:

Lemma 1.1 (Lemma 1.10, [JPS])**:** *Let* V *be a right* R*-module. Then* $V \otimes C$ *is an injective right* C*-comodule. A similar result holds for left* C*-comodules.*

We provide here a more detailed proof than that in the article, in order to explain the action of some morphisms.

Proof: Let (M, ρ^M) be a right C-comodule. Consider the natural transformation:

$$\vartheta_{V,M} = \vartheta : \mathrm{Hom}_R(M, V) \to \mathrm{Hom}^C(M, V \otimes C)$$

defined by the equation:

$$\vartheta(f) = (f \otimes \mathrm{Id}_C) \circ \rho^M, \ \forall f \in \mathrm{Hom}_R(M, V).$$

Also, define

$$\varphi : \mathrm{Hom}^C(M, V \otimes C) \to \mathrm{Hom}_R(M, V), \text{ by } \varphi(g) = (\mathrm{Id}_V \otimes \varepsilon) \circ g, \ \forall g \in \mathrm{Hom}^C(M, V \otimes C),$$

where $\varepsilon : C \to R$ is the counit of C.

We will prove that these maps are mutual inverses. For any $m \in M$, compute:

$$\begin{aligned}
(\varphi\vartheta)(f)(m) &= \varphi((f \otimes \mathrm{Id}_C) \circ \rho^M)(m) \\
&= (\mathrm{Id}_V \otimes \varepsilon) \circ [(f \otimes \mathrm{Id}_C) \circ \rho^M](m) \\
&= (\mathrm{Id}_V \otimes \varepsilon) \circ (f \otimes \mathrm{Id}_C)(\sum m_0 \otimes m_1) \\
&= (\mathrm{Id}_V \otimes \varepsilon)(\sum f(m_0) \otimes m_1) \\
&= \sum f(m_0) \otimes \varepsilon(m_1) \\
&= \sum f(\varepsilon(m_1)m_0) \\
&= f(m).
\end{aligned}$$

For the penultimate equality we have used the fact that the tensor product is R-balanced and for the last one, the counit property for ε.

The other composition computes to:

$$\begin{aligned}
(\vartheta\varphi)(g)(m) &= \vartheta((\mathrm{Id}_V \otimes \varepsilon) \circ g)(m) \\
&= [((\mathrm{id}_V \otimes \varepsilon) \circ g) \otimes \mathrm{Id}_C]\rho^M(m) \\
&= (((\mathrm{Id}_V \otimes \varepsilon) \circ g) \otimes \mathrm{Id}_C)(\sum m_0 \otimes m_1) \\
&= \sum ((\mathrm{Id}_V \otimes \varepsilon) \circ g)(m_0) \otimes m_1 \\
&= \sum (\mathrm{Id}_V \otimes \varepsilon)(g(m_0)) \otimes m_1.
\end{aligned}$$

Now, since g is a comodule map, it follows that it is compatible with the respective structure morphisms, i.e. $(\mathrm{Id}_V \otimes \Delta_C) \circ g = (g \otimes \mathrm{Id}_C) \circ \rho^M$, so for any $m \in m$ with $g(m) = \sum_i v_i \otimes c_i \in V \otimes C$, the following equations must coincide:

$$(\mathrm{Id}_V \otimes \Delta_C)(g)(m) = (\mathrm{Id}_V \otimes \Delta_C)(\sum_i v_i \otimes c_i) = \sum_i \sum v_i \otimes c_{i_1} \otimes c_{i_2};$$

$$(g \otimes \mathrm{Id}_C)\rho^M(m) = (g \otimes \mathrm{Id}_C)(\sum m_0 \otimes m_1) = \sum g(m_0) \otimes m_1.$$

Plugging this into the last equality above gives the following:

$$\begin{aligned}\sum (\mathrm{Id}_V \otimes \varepsilon)(g(m_0)) \otimes m_1 &= \sum_i \sum (\mathrm{Id}_V \otimes \varepsilon)(v_i \otimes c_{i_1} \otimes c_{i_2}) \\ &= \sum_i \sum v_i \otimes \varepsilon(c_{i_1}) \otimes c_{i_2} \\ &= \sum v_i \otimes c_i = g(m).\end{aligned}$$

The last equality follows by the counit property again.

Therefore, the two transformations are isomorphic. Hence, $\mathrm{Hom}^C(-, V \otimes C)$ and $\mathrm{Hom}_R(-, V) \circ U : \mathcal{M}^C \to \mathcal{A}b$ are isomorphic functors, where $U : \mathcal{M}^C \to \mathcal{M}_R$ is the forgetful functor.

Since the base ring R is semisimple, we conclude that $\mathrm{Hom}^C(-, V \otimes C)$ is exact, because $\mathrm{Hom}_R(M, V)$ is so and this makes the comodule $V \otimes C$ injective. □

For bicomodules, a similar result holds.

Lemma 1.2 (Lemma 1.10, [JPS])**:** *If W is an injective R-bimodule, then $C \otimes W \otimes C$ is an injective C-bicomodule.*

Proof: The procedure is similar to that of the proof in the case of lateral comodules. For the bicomodule case, keeping the notations and the context, one uses the functorial isomorphism:

$$\Theta_{M,W} : \mathrm{Hom}_{R-R}(M, W) \to \mathrm{Hom}^{C-C}(M, C \otimes W \otimes C),$$

defined by the equation: $\Theta_{M,W}(f) = (\mathrm{Id}_C \otimes f \otimes \mathrm{Id}_C) \circ (\lambda^M \otimes \mathrm{Id}_C) \circ \rho^M$. □

1.1.3 Graded Linear Duality

Now we turn our attention to a tool which can relate dual structures such as R-rings and R-corings. We will present the graded case, which is used under the assumption of local finiteness, a prerequisite which we make more precise:

Definition 1.5: Let $V = \oplus_{n \in \mathbb{N}} V_n$ be a graded R-bimodule. V is called *left (right) locally finite* if and only if all of its components V_n are finitely generated as left (right) R-modules. When local finiteness is true on both sides, we say that V is *locally finite*.

Throughout this entire section and whenever we refer to graded linear duality, we will assume that all R-(co)rings are locally finite. Note that under the hypothesis of R being a semisimple ring, all R-bimodules are also projective and injective on both sides.

Let us discuss the left dual of an R-bimodule first. Let V be an R-bimodule, we do not need grading for the moment. Define the left dual of V by ${}^*V = \mathrm{Hom}_R({}_RV, {}_RR)$. It becomes a bimodule over the opposite ring R^{op} with respect to the following actions (defined for any $\alpha \in {}^*V,\ r \in R$):

$$(r \cdot \alpha)(v) = \alpha(v)r \qquad \text{and} \qquad (\alpha \cdot r)(v) = \alpha(vr).$$

In the graded case, if $V = \oplus_{n\in\mathbb{N}} V_n$ is a graded R-bimodule, we define the *left graded dual* of V as being the direct sum of component-wise duals, i.e. the R^{op}-bimodule ${}^{*-gr}V = \oplus_{n\in\mathbb{N}} {}^*(V_n)$.

Now, in order to check the algebraic structures that these duals carry, recall that the dual tensor product of two finite dimensional vector spaces is the tensor product of the duals, a property which holds for R-bimodules as well. Concretely, if V, W are R-bimodules, there exists a bi-additive map (defined for arbitrary elements $\alpha \in {}^*V$, $\beta \in {}^*W$ and $v \in V, w \in W$):

$$\phi' : {}^*V \times {}^*W \to {}^*(V \otimes_R W), \qquad \phi'(\alpha, \beta)(v \otimes_R w) = \alpha(v\beta(w))$$

Furthermore, ϕ' behaves well with respect to the left and right action of R on V and W, respectively, as it can be seen from the following computations:

$$\phi'(\alpha \cdot r) \otimes_{R^{op}} \beta)(v \otimes_R w) = (\alpha \cdot r)(v\beta(w)) = \alpha(rv\beta(w)) = \phi'(\alpha \otimes_{R^{op}} (r \cdot \beta))(v \otimes_R w).$$

Therefore, the map ϕ' is R^{op}-balanced, so it induces a morphism of Abelian groups:

$$\phi : {}^*V \otimes_{R^{op}} {}^*W \to {}^*(V \otimes_R W), \qquad \phi(\alpha \otimes_{R^{op}} \beta)(v \otimes_R w) = \alpha(v\beta(w)).$$

Moreover, in fact, ϕ is an R^{op}-bimodule morphism, as it results from the following easy computations:

$$[r \cdot \phi(\alpha \otimes_{R^{op}} \beta)](v \otimes_R w) = \phi(\alpha \otimes_{R^{op}} \beta)(v \otimes_R w)r = \alpha(v\beta(w))r;$$
$$\phi((r \cdot \alpha) \otimes_{R^{op}} \beta)(v \otimes_R w) = (r \cdot \alpha)(v\beta(w)) = \alpha(v\beta(w))r.$$

Right linearity is proved completely analogously.

We claim that when adding the assumption of finite generation of W as a left R-module, the map ϕ becomes a bijection. Indeed, given the semisimplicity of R, we know that W is a projective left R-module. By the dual bases theorem, it follows that there exist finite dual bases on W, that is two sets $\{w_1, \dots, w_n\} \subseteq W$ and $\{{}^*w_1, \dots, {}^*w_n\} \subseteq {}^*W$ such that $w = \sum_{i=1}^n {}^*w_i(w)w_i$, for all elements $w \in W$. Now we define a map which will prove to be the inverse of ϕ. As such, consider the morphism $\psi : {}^*(W \otimes_R W) \to {}^*V \otimes_{R^{op}} {}^*W$, defined by:

$$\psi(\gamma) = \sum_{i=1}^{n} \gamma(- \otimes_R w_i) \otimes_{R^{op}} {}^*w_i.$$

In this expression, $\gamma \in {}^*(V \otimes_R W)$ and the placeholder notation means that in general the map $\gamma(- \otimes_R w_i) : V \to R$ acts as $v \mapsto \gamma(v \otimes_R w_i)$. We are left with proving now that ψ and ϕ are mutual inverses, which will be technical, but straightforward.

Let $\gamma = \phi(\alpha \otimes_{R^{op}} \beta)$ for some $\alpha \in {}^*V,\ \beta \in {}^*W$. Thus, $\gamma(v \otimes_R w) = \alpha(v\beta(w))$, for all elements $v \in V, w \in W$. By the definition of the map ψ above, we compute:

$$\psi(\gamma) = \sum_{i=1}^{n} \gamma(- \otimes_R w_i) \otimes_{R^{op}} {}^*w_i = \sum_{i=1}^{n} \alpha(-\beta(w_i)) \otimes_{R^{op}} {}^*w_i = \sum_{i=1}^{n} \alpha \cdot \beta(w_i) \otimes_{R^{op}} {}^*w_i.$$

Now we can use the definition of the left R-action on ${}^*W$ and also that of dual bases, together with the fact that β is a left R-linear map, so $\beta = \sum_{i=1}^n \beta(w_i) \cdot {}^*w_i$. Therefore, the equality finishes with:

$$\psi(\gamma) = \alpha \otimes_{R^{op}} \left(\sum_{i=1}^{n} \beta(w_i) \cdot {}^*w_i \right) = \alpha \otimes_R \beta,$$

which means that ψ is a left inverse of ϕ. On the other hand, let $\gamma \in {}^*(V \otimes_R W)$, then:

$$[\phi(\psi(\gamma))](v \otimes_R w) = \sum_{i=1}^{n} \phi\Big(\gamma(- \otimes_R w_i) \otimes_{R^{op}} {}^*w_i\Big))(v \otimes_R w) = \sum_{i=1}^{n} \gamma(- \otimes_R w_i)(v^* w_i(w)).$$

We can continue the computations as follows:

$$\sum_{i=1}^{n} \gamma(- \otimes_R w_i)(v^* w_i(w)) = \sum_{i=1}^{n} \gamma(v^* w_i(w) \otimes_R w_i) = \gamma(v \otimes_R \sum_{i=1}^{n} {}^*w_i(w) w_i) = \gamma(v \otimes_R w),$$

which shows that ψ is a right inverse of ϕ as well and finishes the proof of our claim.

Next we study graded (linear) dual for an R-(co)ring, which will be the case in most examples that appear in our study on Koszul (co)rings. We note that the first discussion of the left (right) dual for a left (right) finitely generated R-ring appeared in [BW, §17.9]. This construction can be easily adapted for the case of a left locally finite connected graded R-ring $A = \oplus_{n \in \mathbb{N}} A^n$ and we will do just that. Define ${}^{*-gr}A = \oplus_{n \in \mathbb{N}} {}^*(A^n)$, which becomes an R^{op}-bimodule with respect to the actions defined previously, since an R-ring is also an R-bimodule. When there is no risk of confusion, we will drop the parentheses to avoid unnecessary clutter. As such, instead of ${}^*(A^n)$ we will write simply ${}^*A^n$.

Furthermore, one can make ${}^{*-gr}A$ into a graded connected R^{op}-coring and for this purpose consider the diagram below:

$$\begin{array}{ccc} {}^*A^{n+m} & =\!=\!=\!=\!= & {}^*A^{n+m} \\ {\scriptstyle {}^*\mu_{n,m}} \downarrow & & \vdots\, {\scriptstyle \Delta_{n,m}} \\ {}^*(A^n \otimes_R A^m) & \xrightarrow[\psi]{} & {}^*A^n \otimes_{R^{op}} {}^*A^m \end{array}$$

Here, the leftmost vertical arrow is the transpose of the component for the multiplication of A, namely $\mu^{n,m} : A^n \otimes A^m \to A^{n+m}$. The lower morphism ψ is that which we described above and we can define $\Delta_{n,m} = \psi \circ {}^*\mu_{n,m}$, which is a morphism of R^{op}-bimodules and then the family $\{\Delta_{n,m}\}_{n,m \in \mathbb{N}}$ induces a map $\Delta : {}^{*-gr}A \to {}^{*-gr}A \otimes {}^{*-gr}A$ which respects the gradings on both ${}^{*-gr}A$ and ${}^{*-gr}A \otimes {}^{*-gr}A$.

The situation follows similarly to the construction of the dual coalgebra for an algebra. Let $\alpha \in {}^*A^{n+m}$. One can prove that the relation:

$$\Delta_{n,m}(\alpha) = \sum_{i=1}^{p} \alpha_i' \otimes_{R^{op}} \alpha_i'' \tag{1.6}$$

holds true for some elements $\alpha_1', \ldots, \alpha_p', \alpha_1'', \ldots, \alpha_p'' \in {}^*A$ if and only if the following equation holds true:

$$\alpha(a'a'') = \sum_{i=1}^{p} \alpha_i'(a'\alpha_i''(a'')), \tag{1.7}$$

for all $a' \in A^n$ and $a'' \in A^m$. This makes it easy to see that Δ defines a coassociative comultiplication on ${}^{*-gr}A$, which respects the grading. Moreover, since there is an isomorphism of rings $\mathrm{Hom}_R({}_RR, {}_RR) \simeq R^{op}$, it follows that we can identify ${}^*A^0$ with R^{op} as R^{op}-bimodules. Furthermore, this isomorphism can be extended in a unique way to an R^{op}-bimodule morphism $\varepsilon : {}^{*-gr}A \to R^{op}$ such that it vanishes on all other homogeneous components of ${}^{*-gr}A$. This way $({}^{*-gr}A, \Delta, \varepsilon)$ becomes a graded connected R^{op}-coring, which will be called *the graded left dual* R^{op} *coring of* A or sometimes shorter, *the graded left dual of* A.

For the comultiplication of ${}^{*-gr}A$, we will use a Sweedler-type notation, namely:

$$\Delta_{n,m}(\alpha) = \sum \alpha_{1,n} \otimes \alpha_{2,m}.$$

Using this, the equation (1.7) above can be rewritten as:

$$\alpha(a'a'') = \sum \alpha_{1,n}\big(a'\alpha_{2,m}(a'')\big).$$

As expected, one can discuss about the graded linear dual for an R-coring and obtain an R^{op}-ring. As such, let $C = \oplus_{n\in\mathbb{N}} C_n$ be a graded connected R-coring. We define the graded left dual of C by ${}^{*-gr}C = \oplus_{n\in\mathbb{N}} {}^*C_n$, where ${}^*C_n = \mathrm{Hom}_R({}_RC, {}_RR)$, as in the previous case. Now, ${}^{*-gr}C$ becomes an R^{op}-ring using the *graded convolution product* which, for $\alpha \in {}^*C_n$ and $\beta \in {}^*C_m$ is given by:

$$\alpha * \beta = \mu^{n,m}(\alpha \otimes \beta) \iff (\alpha * \beta)(c) = \sum \alpha\big(c_{1,n}\beta(c_{2,m})\big),\ \forall c \in C_{n,m}.$$

The unit for the R^{op}-ring ${}^{*-gr}C$ coincides with the counit of C. A remark which we must make is that, as in the case of the dual algebra associated to an algebra, the graded (left) dual makes sense for a graded R-coring even when dropping the assumption of local finiteness.

In a very similar way one can define and construct the graded right duals, both in the case of R-rings and in the case of R-corings, which we will skip under the remark of similitude.

We close this subsection adding the fact that graded linear duality is, indeed, a categorical duality, in the sense that for a locally finite R-ring A, there are canonical isomorphisms $({}^{*-gr}A)^{*-gr} \simeq A \simeq {}^{*-gr}(A^{*-gr})$ and the corresponding ones for R-corings C, which one easily obtains by somewhat tedious, but easy computations.

1.1.4 The Normalised (Co)Chain Complex

For a graded connected R-ring A it is of special interest in the study of Koszulity to compute and describe $T(A) = \oplus_{n\in\mathbb{N}} \mathrm{Tor}_n^A(R, R)$. This will be obtained as the nth homology group of the normalised bar complex $(\Omega_\bullet(A), d_\bullet)$. The components of this complex are defined by $\Omega_n(A) = A_+^n$ and the differentials $d_n : \Omega_n(A) \to \Omega_{n-1}(A)$ are defined by $d_1 = 0$ and for all $n > 1$, by the equation:

$$d_n(a_1 \otimes a_2 \otimes \ldots \otimes a_n) = \sum_{i=1}^{n-1} (-1)^{i-1} a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n.$$

This normalised bar complex has a canonical structure of a differential graded (DG, for short) coalgebra in the category of R-bimodules, with respect to the comultiplication given by the cap product:

$$\Delta_{p,q}(a_1 \otimes \ldots \otimes a_{p+q}) = (a_1 \otimes \ldots \otimes a_p) \otimes (a_{p+1} \otimes \ldots \otimes a_{p+q}),$$

therefore $T(A) = \oplus_{n \in \mathbb{N}} T_n(A)$ has a canonical structure of a connected R-ring.

We now go into some technical details which will be used in the proofs of subsequent results. It will be useful to remark that the normalised bar complex $\Omega_\bullet(A)$ decomposes as a direct sum of subcomplexes $\Omega_\bullet(A) = \oplus_{m \geq 0} \Omega_\bullet(A, m)$. These summands are defined as follows. Let us call an n-tuple $\mathfrak{m} = (m_1, \ldots, m_n)$ a *positive n-partition of* m if and only if the components are positive and add to m, that is $\sum_{i=1}^n m_i = m$. Denote the set of all positive n-partitions of m by $\mathfrak{P}_n(m)$. We will use another shorthand notation: if $\mathfrak{m} \in \mathfrak{P}_n(m)$ and A is a graded connected R-ring, then the tensor product $A^{m_1} \otimes A^{m_2} \otimes \ldots \otimes A^{m_n}$ will be denoted simply by $A^{\mathfrak{m}}$. Also, in the former context, the multiplication μ of A induces bimodule maps $\mu^{\mathfrak{m}} : A^{\mathfrak{m}} \to A^m$. We can now describe the component $\Omega_\bullet(A, m)$ as being the following subcomplex of $\Omega(A)$:

$$0 \xleftarrow{d_0^m} 0 \xleftarrow{d_1^m} \bigoplus_{\mathfrak{m}_1 \in \mathfrak{P}_1(m)} A^{\mathfrak{m}_1} \xleftarrow{d_2^m} \bigoplus_{\mathfrak{m}_2 \in \mathfrak{P}_2(m)} A^{\mathfrak{m}_2} \xleftarrow{d_3^m} \ldots \xleftarrow{d_n^m} \bigoplus_{\mathfrak{m}_n \in \mathfrak{P}_n(m)} A^{\mathfrak{m}_n} \leftarrow \ldots. \quad (1.8)$$

It can be seen that, since there is only one 1-partition of m, namely $\mathfrak{m}_1 = (m)$, it follows that the first direct sum in $\Omega_\bullet(A, m)$ coincides with A^m. For this subcomplex, the homology in degree n will be denoted by $T_{n,m}(A)$ or $\mathrm{Tor}^A_{n,m}(R, R)$. This is a bigraded coring and one can define $T(A) = \oplus_{m \geq 0} T_{n,m}(A)$, a decomposition which is compatible with the coring structure.

A completely similar construction can be performed by duality for an R-coring C. The normalised bar cochain complex $(\Omega^\bullet(C), d^\bullet)$, which has the components given by $\Omega^n(C) = C_+^{(n)}$, $d^0 = 0$ and in general:

$$d^n = \sum_{i=1}^{n} (-1)^{i-1} \mathrm{Id}_{C_+^{(i-1)}} \otimes \Delta_+ \otimes \mathrm{Id}_{C_+^{(n-i)}}.$$

Furthermore, $E^n(C) = \mathrm{Ext}^n_C(R, R)$ is the nth cohomology group of this complex and $E(C) = \bigoplus_{n \in \mathbb{N}} E^n(C)$ is a connected R-ring with respect to the multiplication induced by the DG algebra structure on $\Omega^\bullet(C)$, defined by the concatenation of tensor monomials.

There is a similar decomposition as a direct sum of subcomplexes $\Omega^\bullet(C, m)$, defined as follows. Denote by $C_{\mathfrak{m}} = C_{m_1} \otimes \ldots \otimes C_{m_n}$ for $\mathfrak{m} \in \mathfrak{P}_n(m)$. Then $\Omega^\bullet(C, m)$ is the subcomplex below:

$$0 \to 0 \xrightarrow{d_m^0} \bigoplus_{m_1 \in \mathfrak{P}_1(m)} C_{m_1} \xrightarrow{d_m^1} \bigoplus_{m_2 \in \mathfrak{P}_2(m)} C_{m_2} \xrightarrow{d_m^2} \dots \xrightarrow{d_m^{n-1}} \bigoplus_{m_n \in \mathfrak{P}_n(m)} C_{m_n} \xrightarrow{d_m^n} \dots .$$

Similarly, we remark that $\Omega^1(C, m) = C_m$ and the homology in degree n of $\Omega^\bullet(C, m)$ will be denoted by $E^{n,m}(C)$ or $\mathrm{Ext}_C^{n,m}(R, R)$. Also, there is a decomposition of $E(C)$ in the form $E(C) = \oplus_{m \geq 0} E^{n,m}(C)$, which is compatible with the ring structure in the sense that $E(C)$ is a bigraded ring having the homogeneous component of bidegree (n, m) exactly $E^{n,m}(C)$. Further details of these constructions and their properties are found in [JPS, §1.15].

It is now useful to prove a lemma in a general context regarding the injectivity of a morphism between two corings, a result that extends a known one, in the case of coalgebras.

Lemma 1.3: *Let C be a connected R-coring. The following results hold true.*

(1) C is strongly graded if and only if $PC = C_1$;

(2) If C is strongly graded and $f : C \to D$ is a morphism of graded R-corings such that the components f_0 and f_1 are injective, then f is injective;

(3) Let $C = \bigoplus_{n,m \geq 0} C_{n,m}$ be a bigraded R-coring. If $\mathrm{gr}(C)$ is strongly graded and $C_{n,m} = 0$ for $n = 0, 1$ and all $m \neq n$, then the component $C_{n,m}$ is also null, for all $n \geq 2$ and $m \neq n$.

Proof: (1) For the first part, assume C is strongly graded. Let $x = \sum_{k=0}^{n} x_k$ be a primitive element of C, where $x_k \in C_k$ for all k. Since C is graded, its comultiplication can be decomposed as $\Delta = \sum_{p=0}^{k} \Delta_{p,k-p}$. This, along with the primitivity of x gives the following:

$$\sum_{k=0}^{n} x_k \otimes 1 + \sum_{k=0}^{n} 1 \otimes x_k = x \otimes 1 + 1 \otimes x = \Delta(x) = \sum_{k=0}^{n} \sum_{p=0}^{k} \Delta_{p,k-p}(x_k).$$

We claim that $x_k = 0$, for all $k \neq 1$. Since $x_0 \in C_0$, we have $\Delta(x_0) = \Delta_{0,0}(x_0) = x_0 \otimes 1$, hence by equating the terms in $C_0 \otimes C_0$ it follows that $x_0 \otimes 1$, which implies $x_0 = 0$. Let us fix now $k \geq 2$. The rightmost term of the equation above $\Delta_{p,k-p}(x_k)$ is an element of $C_p \otimes C_{k-p}$. On the other hand, $x_k \otimes 1 + 1 \otimes x_k \in C_k \otimes C_0 + C_0 \otimes C_k$. It follows that $x_k = 0$, which proves the claim. The conclusion is that $x = x_1 \in C_1$ and hence $PC \subseteq C_1$. The reverse inclusion holds in general, which completes the proof of the direct implication.

For the converse, assume $PC = C_1$ and we will prove that all components of the comultiplication of C are injective. We first remark that

$$\mathrm{Ker}\Delta(2) = \mathrm{Ker}\Delta_{1,1} = PC \cap C_2 = 0,$$

so this map is injective. Assume for an inductive argument that $\Delta(k)$ is injective for all $k \leq n$. Therefore, using equation (1.3) this is equivalent to $\Delta(p) \otimes \Delta(n+1-p)$ being injective for all $0 < p < n+1$. Further, by the same equation, this implies that

$\Delta_{p,n+1-p}(x) = 0$ for any $x \in \mathrm{Ker}\Delta(n+1)$ and all p as above. Thus, $x \in PC \cap C_{n+1} = 0$, which makes $\Delta(n+1)$ injective, as required.

(2) We will prove the second assertion by induction. Assume that the component f_k is injective for all $k = 0, \dots, n$. Since f is a morphism of graded corings, it makes the diagram below commutative:

$$\begin{array}{ccc} C_{n+1} & \xrightarrow{\Delta^C_{n,1}} & C_n \otimes C_1 \\ {\scriptstyle f_{n+1}}\downarrow & & \downarrow{\scriptstyle f_n \otimes f_1} \\ D_{n+1} & \xrightarrow{\Delta^D_{n,1}} & D_n \otimes D_1. \end{array}$$

As we assumed that $\Delta^C_{n,1}$ and $f_n \otimes f_1$ are injective, it follows that f_{n+1} is injective as well. Therefore, by induction, all components of f are injective, so the whole map f is so.

(3) For the final part of the lemma, let $\mathrm{Diag}(C) = \oplus_{n \geq 0} C_{n,n}$ and also the graded map $\mathrm{gr}(\pi) : \mathrm{gr}(C) \to \mathrm{Diag}(C)$ be the canonical morphism of graded R-corings introduced in the previous section. Since $\mathrm{gr}_1(C) = \oplus_m C_{1,m} = C_{1,1}$, the component $\mathrm{gr}_1(\pi)$ coincides with the identity map of $C_{1,1}$, which, in turn, coincides with $P(\mathrm{gr}(C)) = \mathrm{Diag}_1(C)$. On the other hand, since $C_{0,0} = \mathrm{gr}_0(C) = \mathrm{Diag}_0(C)$, the component $\mathrm{gr}_0(\pi)$ coincides with the identity map of $C_{0,0}$. Now using the first part of the lemma, we get that $\mathrm{gr}(\pi)$ is injective and as $\mathrm{Kergr}_n(\pi) = \oplus_{m \neq n} C_{n,m}$, it follows that $C_{n,m} = 0$, for all $n \neq m$. This completes the proof. □

Remark 1.1: A similar result in the case of ordinary $\mathbb{k}$-coalgebras ($\mathbb{k}$ being a field) was proved in [MiS, Lemma 2.3].

As expected, there is a dual result of the lemma, which we now state and prove.

Lemma 1.4: *Let A be a graded connected R-ring. The following hold true.*

(1) A is strongly graded if and only if the canonical map $QA \to A^1$ is injective if and only if the canonical map $A^1 \to QA$ is surjective.

(2) If A is strongly graded, let B be a connected graded R-ring and $g : B \to A$ a morphism of graded R-rings such that the first components g^0 and g^1 are surjective. Then g is surjective.

(3) Assume that A is bigraded, $A = \oplus_{m,n \geq 0} A^{n,m}$. If $\mathrm{gr}(A)$ is strongly graded and $A^{n,m} = 0$ for $n = 0, 1$ and all $m \neq n$, then $A^{n,m} = 0$ for all $2 \leq n \neq m$.

Proof: (1) The map $QA \to A^1$ is a left inverse of $A^1 \to QA$, so surjectivity of the latter is equivalent to injectivity of the former. Further, A being strongly graded is equivalent to the inclusion $A_n \subseteq A_+^2$, for all $n \geq 2$. Therefore, the first statement holds true, since the inclusions are true if and only if the map $A^1 \to QA$ is surjective.

(2) The proof of the second part is dual to that of Lemma 1.3(2), by means of the diagram below.

$$\begin{array}{ccc} B_n \otimes B_1 & \xrightarrow{\mu_B^{n,1}} & B_{n+1} \\ {\scriptstyle g^n \otimes g^1}\downarrow & & \downarrow{\scriptstyle g^{n+1}} \\ A_n \otimes A_1 & \xrightarrow{\mu_A^{n,1}} & A_{n+1} \end{array}$$

(3) For this last part of the lemma, define $\mathrm{Diag}(A)$ as the graded subring of $\mathrm{gr}(A)$ given by $\mathrm{Diag}(A) = \oplus_{n\in\mathbb{N}} A^{n,n}$. Let $i : \mathrm{Diag}(A) \hookrightarrow \mathrm{gr}(A)$ denote the inclusion. By the hypothesis, i^0 and i^1 are surjective. Hence, using the second part of the lemma, i is surjective, which in turn implies that $A^{n,m} = 0$, $\forall n \neq m$. □

1.2 (Almost) Koszul Pairs

We now pass to the introduction of the main tools of study used in the remainder of the thesis. Following [JPS], we define and discuss fundamental properties and examples for (almost) Koszul pairs, making use of the general notions introduced thus far.

In [JPS], the authors provided a new tool for studying Koszul R-rings, by means of a pair consisting of an R-ring and an R-coring, connected by an R-bimodule isomorphism. They associated to this pair six complexes which extend the known *Koszul complexes* and proved that if any of those complexes is exact, all of them are so. In this situation, the pair was called a *Koszul pair* and it was later proved that similar to the classical case, the Koszul pair consists of a Koszul R-ring (in the sense of [BGS]) and a Koszul coring. This latter structure was introduced and studied in [MS1].

We will now recollect the basic constructions regarding (almost) Koszul pairs and also provide some first examples of this type of structures. Later, in the following chapter, we will present how the notions can be dualised in order to obtain *Koszul corings* and lastly, in the third chapter, we will provide further examples related to incidence rings for finite, graded posets and monoid rings, respectively.

1.2.1 Almost Koszul Pairs

We will work with an R-ring A and an R-coring C, both of which are assumed to be graded and connected and the base ring R is taken to be semisimple. Unless otherwise specified, when we refer to either A, C or R they will have this meaning. Also, we will stick to the notations and conventions mentioned in the previous section.

By [JPS, §1.3], the definition of *almost Koszul pairs* is the following:

Definition 1.6: The pair (A, C) is called *almost Koszul* if there exists an isomorphism of R-bimodules $\theta = \theta_{C,A} : C_1 \to A^1$ such that the following composition is the zero map:

$$C_2 \xrightarrow{\Delta_{1,1}} C_1 \otimes C_1 \xrightarrow{\theta\otimes\theta} A^1 \otimes A^1 \xrightarrow{\mu^{1,1}} A^2. \tag{1.9}$$

Using Sweedler's notation for the graded case (see equation (1.1)), the nullity of the composition above can be written as:

$$\sum \theta(c_{1,1})\theta(c_{2,1}) = 0,\ \forall c \in C_2. \tag{1.10}$$

It is useful to abstract this notion into a category as follows. Let (A, C) and (B, D) be almost Koszul pairs. Then a morphism of almost Koszul pairs is a couple (ϕ, ψ) of maps, where $\phi : A \to B$ is a morphism of graded connected R-rings and $\psi : C \to D$ is a morphism of graded connected R-corings such that they commute with the structural isomorphisms $\theta_{C,A}$ and $\theta_{B,D}$ respectively. More precisely, the diagram below is commutative:

$$\begin{array}{ccc} A^1 & \xrightarrow{\phi_1} & B^1 \\ \downarrow{\scriptstyle \theta_{A,C}} & & \downarrow{\scriptstyle \theta_{B,D}} \\ C_1 & \xrightarrow{\psi_1} & D_1 \end{array}$$

A first example of an almost Koszul pair is provided in [JPS, Proposition 1.8] and we recall it here. For any connected and strongly graded R-ring A, the pair $(A, T(A))$ is almost Koszul. The coring structure of $T(A)$ is obtained using the normalised bar complex (see §§1.1.4) and note that the structural map $\theta_{A,T(A)} : T_1(A) \to A^1$ is induced by the projection $A_+ \to A^1$. This is an isomorphism, because since A is strongly graded, $T_1(A) = A_+/A_+^2 \simeq A^1$.

Using the shriek coring we can provide yet another example of an almost Koszul pair, namely $(A, A^!)$. Recall that since $A^!_1 = A^1$, the structural isomorphism can be taken as the identity of A^1. Furthermore, the condition (1.10) is trivially satisfied, because $A^!_2 = \mathrm{Ker}\mu^{1,1}$, by construction (see Definition 1.3).

By duality, one obtains other two almost Koszul pairs, namely $(E(C), C)$ and $(C^!, C)$. If C is connected and strongly graded, then [JPS, Proposition 1.18] proves exactly this, that the pair $(E(C), C)$ is almost Koszul. The R-ring structure of $E(C)$ is defined using the normalised cobar complex, as in §§1.1.4. Since $E^1(C) = \mathrm{Ker}\Delta_+$, in order to obtain the structural isomorphism, we put $\theta_{E(C),C} : C_1 \to E^1(C)$, which is defined by $\theta_{E(C),C}(c) = c + C_0 \in C/C_0$. This map is well defined and is an isomorphism of R-bimodules.

For the second pair, recall that $C^!_1 = C_1$ and further that $C^!_2 = (C_1 \otimes C_1)/\mathrm{Im}\Delta_{1,1}$, so taking the structural map as being the identity of C_1 we get the required isomorphism. Furthermore, the equation (1.10) is verified, since the $(1,1)$ component of the multiplication of C^1 coincides with the canonical projection $C_1 \otimes C_1 \to C^!_2$.

In characterising Koszulity for an almost Koszul pair we will need to introduce six complexes. They will pertain to different categories, left and right A-modules and A-bimodules plus for the dual version, left and right C-comodules and C-bicomodules respectively. They play the role of the classical Koszul complex for a Koszul algebra (as well for coalgebras, as we will see) and also provide appropriate projective and injective resolutions of R in the respective categories. Therefore, let us continue with just this, a recollection of the six complexes.

Let (A, C) be an almost Koszul pair. A complex of *graded right* C-*comodules* could be obtained as follows. Define:

$$K_r^{-1}(A, C) = R \qquad \text{and} \qquad K_r^n(A, C) = A^n \otimes C, \ \forall n \geq 0. \tag{1.11}$$

The differential map of degree +1 $d_r^n : A^n \otimes C \to A^{n+1} \otimes C$ is zero on $A^n \otimes C_0$ and in rest, for all $p > 0$ and $a \otimes c \in A^n \otimes C_p$, we put:

$$d_r^n(a \otimes c) = \sum a\theta(c_{1,1}) \otimes c_{2,p-1}. \tag{1.12}$$

In this equation, $\theta = \theta_{C,A} : C_1 \to A^1$ is the structural R-bimodule isomorphism of the almost Koszul pair (A, C) and in a more expanded form, the action of the differential is, in fact, the following composition:

$$A^n \otimes C_p \xrightarrow{A^n \otimes \Delta_{1,p-1}} A^n \otimes C_1 \otimes C_{p-1} \xrightarrow{A^n \otimes \theta \otimes C_{p-1}} A^n \otimes A^1 \otimes C_{p-1} \xrightarrow{\mu^{n,1} \otimes C_{p-1}} A^{n+1} \otimes C_{p-1}.$$

One could work over the opposite pair as well. Indeed, by [JPS, Remark 1.4], since we know that A^{op} and C^{op} are R^{op}-rings (and corings, respectively), (A^{op}, C^{op}) is an almost Koszul pair over R^{op} if and only if (A, C) is an almost Koszul pair over R. The structural isomorphisms coincide as maps, but their action is adapted to the respective category, of R-bimodules or of R^{op}-bimodules respectively.

Thus, in this situation, one could readily obtain another complex, in the category of left C-comodules working with the opposite pair. As such, define the complex $K_l^\bullet(A, C) = K_r^\bullet(A^{op}, C^{op})$ and accordingly, d_l^n to be the zero map on $A^n \otimes C_0$ and:

$$d_l^n(a \otimes c) = \sum a\theta(c_{1,1}) \otimes c_{2,p-1},$$

for all $p > 0$ and $a \otimes c \in A^n \otimes C_p$.

Using these definitions, we have the expected result:

Proposition 1.1 (Proposition 1.12, [JPS]): *If (A, C) is an almost Koszul pair, then the complexes $(K_l^\bullet(A, C), d_l^\bullet)$ and $(K_r^\bullet(A, C), d_r^\bullet)$ are cochain complexes of graded C-comodules (on the left and on the right, respectively).*

In order to obtain the complex of C-bicomodules, one can combine the two lateral complexes as follows. Let $K^{-1}(A, C) = C$ and $K_n(A, C) = C \otimes A^n \otimes C$. For the differentials, let d^{-1} coincide with the comultiplication Δ of C and for $n \geq 0$, define:

$$d^n = d_l^n \otimes \mathrm{Id}_C + (-1)^{n+1}\mathrm{Id}_C \otimes d_r^n.$$

It follows immediately that this complex is indeed in the category of C-bicomodules.

For the other three complexes in the categories of left, right and bicomodules over A, we will be brief, as they are obtained analogously. Therefore, one defines three chain complexes $K_\bullet(A, C)$, $K_\bullet^r(A, C)$ and $K_\bullet(A, C)$. The left A-module complex is defined as:

$$K_{-1}^l(A, C) = R \qquad \text{and} \qquad K_n^l(A, C) = A \otimes C_n,$$

the differential d_0^l being the left action of A on $R = C_0$ and for $n > 0$:

$$d_n^l(a \otimes c) = \sum a\theta(c_{1,1}) \otimes c_{2,n-1},$$

for any $a \otimes c \in K_n^l(A, C)$ and θ being the structural isomorphism of the almost Koszul pair.

Working with the opposite pair, one obtains the complex of right A-modules and note that both of the lateral complexes decompose as a direct sum $\oplus_{m\geq 0} K_\bullet^l(A, C, m)$ (and $\oplus_{p\geq 0} K_\bullet^r(A, C, p)$ respectively), where $K_n^l(A, C, m) = A^{m-n} \otimes C_n$ and the appropriate formula for the right version. Of course, for $n > m$ (and $n > p$, respectively), all of the subcomplexes are null.

The complex of A-bimodules has the components $K_{-1}(A, C) = A$ and in rest we have $K_n(A, C) = A \otimes C_n \otimes A$ for all $n \geq 0$ while the map d_0 is induced by the multiplication of A, and for $n > 0$ we have:

$$d_n = d_n^l \otimes \mathrm{Id}_A + (-1)^n \mathrm{Id}_A \otimes d_n^r.$$

There exist some canonical morphisms of almost Koszul pairs which will provide useful in the results that characterise Koszul (co)rings that we will state later on. Let

$(A, A^!)$ and $(A, T(A))$ be the two almost Koszul pairs that we introduced. The first component of the morphism between these pairs is, of course, the identity of A. For the second component, we proceed as in [JPS, Proposition 1.24]. As such, let $\beta^l_\bullet(A)$ be the bar resolution of A. Delete the component of degree -1 and apply the functor $R \otimes_A (-)$ to the morphism $\phi_\bullet : K^l_\bullet(A, A^!) \to \beta^l_\bullet(A)$ from the cited result. Recall that for $a \otimes c \in K^l_n(A, A^!)$, this morphism acts as:

$$\phi_n(a \otimes c) = \sum a \otimes \theta(c_{1,1}) \otimes \theta(c_{2,1}) \otimes \ldots \otimes \theta(c_{n,1}).$$

After applying the mentioned functor, we get a morphism from $R \otimes_A K^l_\bullet(A, A^!)$ to the complex $R \otimes_A \beta^l_\bullet(A)$, both of which are concentrated in non-negative degrees. [JPS, Proposition 1.23] shows that the former complex is isomorphic to $(A^!, 0)$, the trivial complex with zero differentials which has the module of n-chains identical to $A^!_n$. On the other hand, the second complex is exactly $\Omega_\bullet(A)$, by definition. And furthermore, since the map $\mathrm{Id}_R \otimes \phi_\bullet$ is compatible with the coring structures of both $A^!$ and $\Omega_\bullet(A)$, the homology functor will provide the desired coring map $\phi^A : A^! \to T(A)$.

To better explain the action of this map, recall the n-th degree component of $A^!$:

$$A^!_n = \bigcap_{i=1}^{n-1} (A^1)^{i-1} \otimes \mathrm{Ker}\mu^{1,1} \otimes (A^1)^{n-i-1} \subseteq A_+^{(n)},$$

so in fact any element $x \in A^!_n$ is an n-cycle in $\Omega_\bullet(A)$, therefore its image through ϕ^A being its homology class. With this at hand, it is now easy to check that (Id_A, ϕ^A) is a morphism of almost Koszul pairs, as required.

A totally similar argument could be formulated for the dual case. Hence, there is a morphism of almost Koszul pairs (ϕ_C, Id_C) from $(E(C), C)$ to $(C^!, C)$. This is obtained by deleting the component of degree -1 in the respective complexes, then applying the functor $\mathrm{Hom}^C(R, -)$ to the map $\phi^\bullet : \beta^\bullet_r(C) \to K^\bullet_r(C^!, C)$. The dual versions of [JPS, Proposition 1.23, 1.24] provide us with a morphism $\overline{\phi}_\bullet : \Omega^\bullet(C) \to C^!$, the latter having a zero differential. We can now define a coring map $\phi_C : E(C) \to C^!$ by applying the cohomology functor to this morphism. Hence, for any elements $x_1, \ldots, x_n \in C_+$, we have:

$$\overline{\phi}^n(x_1 \otimes \ldots \otimes x_n) = a_1 a_2 \cdots a_n \in C^!_n,$$

where $a_i = (\theta_{C^!,C} \circ \overline{\pi}_1)(x_i) \in C^!_1$ and $\overline{\pi}_1 : C_+ \to C_1$ is induced by the projection. Thence, ϕ_C maps the cohomology class of an n-cocycle ω to $\overline{\phi}^n(\omega)$.

In the following section we will introduce Koszul pairs using exactness of any of six complexes. However, in the weaker almost Koszul case, under a mild assumption, we still have the following result:

Lemma 1.5 (Lemma 2.1, [JPS]): *Let (A, C) be an almost Koszul pair.*

(1) If A is a strongly graded R-ring, then $H_0(K_\bullet(A, C)) = 0$.

(2) If C is a strongly graded R-coring, then $H^0(K^\bullet(A, C)) = 0$.

1.2.2 Koszul Pairs

One of the most important results in [JPS], which justifies the use of the six complexes above and is at the core of this approach of Koszul rings via Koszul pairs is that which relates the exactness of the six complexes constructed in the previous section. Concretely, the result (which also serves as a definition) is as follows.

Theorem 1.1 (Theorem 2.3, [JPS]): *If any of the six complexes from the previous section is exact, then all of them are so. When this is the case, the underlying almost Koszul pair* (A, C) *is called a* Koszul pair.

Moreover, the use of the six complexes is even more justified, since they could be considered *Koszul complexes*, in fact, as the result below shows.

Corollary 1.1 (Corollary 2.5, [JPS]): *Let* (A, C) *be a Koszul pair. The following statements hold true:*

(1) The complex $K_l^\bullet(A, C)$ *is a resolution of* R *by injective graded left* C*-comodules.*

(2) The complex $K_r^\bullet(A, C)$ *is a resolution of* R *by injective graded right* C*-comodules.*

(3) The complex $K_\bullet^l(A, C)$ *is a resolution of* R *by projective graded left* A*-modules.*

(4) The complex $K_\bullet^r(A, C)$ *is a resolution of* R *by projective graded right* A*-modules.*

(5) If A *is an injective* R*-bimodule, then* $K^\bullet(A, C)$ *is a resolution of* C *by injective graded* C*-bicomodules.*

(6) If C *is a projective* R*-bimodule, then* $K_\bullet(A, C)$ *is a resolution of* A *by projective graded* A*-bimodules.*

As per [JPS, Remark 2.6], the last two conditions hold also in the situation when R is a separable algebra over a field $\Bbbk$ (equivalently, R is a projective R-bimodule).

Moreover, in the case when the pair (A, C) is Koszul, one can formulate more powerful results regarding the pairs $(A, T(A))$ and $(E(C), C)$, respectively.

Theorem 1.2 (Theorem 2.9, [JPS]): *Let* (A, C) *be a Koszul pair. The graded* R*-coring* $T(A)$ *is isomorphic to* C *(plus* $E(C)$ *and* A *are isomorphic graded* R*-rings) and* $(A, T(A))$ *and* $(E(C), C)$ *are Koszul pairs.*

Moreover, the "match" to an R-ring or an R-coring which completes the Koszul pair is unique up to an isomorphism ([JPS, Corollary 2.10]) and, in fact, the two pairs in the theorem above are very important since they provide a kind of duality:

Corollary 1.2 (Corollary 2.11, [JPS]): *If* (A, C) *is a Koszul pair, then there exist canonical isomorphisms* $E(T(A)) \simeq A$ *and* $T(E(C)) \simeq C$.

Later on, we will see some related results in the case of the graded linear dual of a locally finite R-(co)ring and we will also put in use the Ext ring of a Koszul ring (somewhat similar to the well-known Yoneda structure) to extend the characterisation of them.

We close this section with recalling the result which shows that indeed the construction of Koszul pairs agrees with the one that [BGS] provided. Furthermore, it implies that Koszul pairs are a good tool for studying Koszul rings, but also opens the way to investigating Koszul corings, both of the tasks being approached in the subsequent chapter.

Theorem 1.3 (Theorem 2.13, [JPS]): *A connected* R*-ring* A *is Koszul (in the sense of [BGS, Definition 1.1.2]) if and only if* A *is strongly graded and* $(A, T(A))$ *is a Koszul pair if and only if there exists a graded* R*-coring* C *such that the pair* (A, C) *is Koszul.*

This result will be the starting point to our further exploration of the equivalent characterisations that one could formulate for Koszul rings, which we will also dualise in order to introduce *Koszul corings*.

Let us now present two special cases for Koszul pairs, which are more complicated. These include twisted tensor products of algebras, for which we will show how an almost Koszul pair could be associated to it and that of braided bialgebras, for which a Koszul pair can be constructed. The presentation will follow the results proved in [JPS, §4, §6].

1.3 Twisting and Entwining Maps

We now present some general facts about twisted tensor products of algebras, as well as results which show how one could associate an almost Koszul pair to a twisted tensor product.

The basic setup is the following. Let A, B be two connected, strongly graded R-rings.

Definition 1.7: A *twisting map* between A and B is an R-bilinear map $\sigma : B \otimes A \to A \otimes B$ which is compatible with the multiplications, i.e.:

$$\sigma \circ (\mathrm{Id}_B \otimes m_A) = (m_A \otimes \mathrm{Id}_B) \circ (\mathrm{Id}_A \otimes \sigma) \circ (\sigma \otimes \mathrm{Id}_A),$$
$$\sigma \circ (m_B \otimes \mathrm{Id}_A) = (\mathrm{Id}_A \otimes m_B) \circ (\sigma \otimes \mathrm{Id}_B) \circ (\mathrm{Id}_B \otimes \sigma).$$

Moreover, σ must be compatible with the units of A and B. That is, $\sigma(1_B \otimes a) = a \otimes 1_B$ and $\sigma(b \otimes 1_A) = 1_A \otimes b$, for all elements $a \in A, b \in B$.

There is a commonly used notation, of a Sweedler type, $\sigma(b \otimes a) = \sum a_\sigma \otimes b_\sigma$. For instance:

$$[(m_A \otimes \mathrm{Id}_B) \circ (\mathrm{Id}_A \otimes \sigma) \circ (\sigma \otimes \mathrm{Id}_B)](b \otimes a' \otimes a'') = \sum a'_\sigma a''_\sigma \otimes ((b_\sigma))_{\sigma'}.$$

The occurrence of σ and σ' shows that the twisting map was applied twice.

Now, in the particular case when A and B are R-rings, their twisted tensor product, which we denote by $A \otimes_\sigma B$ has a canonical structure of an R-ring, with respect to the multiplication:

$$(a' \otimes b')(a'' \otimes b'') = \sum a' a''_\sigma \otimes b'_\sigma b''.$$

The unit for this R-ring is $1_A \otimes 1_B$.

Furthermore, if both A and B are graded, we ask for the twisting map to be compatible with the gradings, that is, to be graded itself. . This means that $\sigma(B^p \otimes A^q)$ is a subset of $A^q \otimes B^p$, for all positive integers p, q. The restriction of σ to $B^p \otimes A^q$ will be denoted by $\sigma^{p,q}$. Therefore, $A \otimes_\sigma B$ becomes a graded R-ring, the homogeneous component of degree n being the direct sum of all R-bimodules $A^p \otimes B^q$ such that $p + q = n$.

Such constructions could be applied to various cases, of which we present only (co)chain R-(co)rings and entwining maps. All of the details are found in [JPS, §4], we will restrict only to the definitions and results which will allow us to introduce an almost Koszul pair consisting of twisted tensor products of R-rings.

Let $(\Omega^\bullet, d^\bullet_\Omega)$ and $(\Gamma^\bullet, d^\bullet_\Gamma)$ be two cochain R-rings. A graded twisting map denoted by $\sigma^{\bullet,\bullet} : \Gamma^\bullet \otimes \Omega^\bullet \to \Omega^\bullet \otimes \Gamma^\bullet$ is called a *twisting map of cochain* R*-rings* if it is compatible with the differentials. That is, it induces the following maps of complexes:

$$\sigma^{\bullet,q} : (\Gamma^\bullet \otimes \Omega^q, d^\bullet_\Gamma \otimes \mathrm{Id}_{\Omega^q}) \to (\Omega^q \otimes \Gamma^\bullet, \mathrm{Id}_{\Omega^q} \otimes d^\bullet_\Gamma)$$

and, symmetrically:

$$\sigma^{p,\bullet} : (\Gamma^p \otimes \Omega^\bullet, \mathrm{Id}_{\Gamma^p} \otimes d^\bullet_\Omega) \to (\Omega^\bullet \otimes \Gamma^q, d^\bullet_\Omega \otimes \mathrm{Id}_{\Gamma^p}),$$

for all positive integers p, q.

As expected, these twisting maps descend well to cohomology.

Proposition 1.2: ([JPS, Proposition 4.3]) *Let* $(\Omega^\bullet, d^\bullet_\Omega)$ *and* $(\Gamma^\bullet, d^\bullet_\Gamma)$ *be two cochain* R*-rings and* V *be an* R*-bimodule.*

(1) if $\varphi^\bullet : (V \otimes \Omega^\bullet, \mathrm{Id}_V \otimes d^\bullet_\Omega) \to (\Omega^\bullet \otimes V, d^\bullet_\Omega \otimes \mathrm{Id}_V)$ *is a morphism of complexes which is compatible with the unit and the multiplication of* $\Omega^\bullet$*, then* $\overline{\varphi}^\bullet = H^\bullet(\varphi^\bullet)$ *is compatible with the unit of the graded* R*-ring* $H^\bullet(\Omega^\bullet)$*.*

(2) If $\chi^\bullet : (\Gamma^\bullet \otimes V, d^\bullet_\gamma \otimes \mathrm{Id}_V) \to (V \otimes \Gamma^\bullet, \mathrm{Id}_V \otimes d^\bullet_\Gamma)$ *is a morphism of complexes that is compatible with the unit and the multiplication of* $\Gamma^\bullet$*, then* $\overline{\chi}^\bullet = H^\bullet(\chi^\bullet)$ *is compatible with the multiplication and the unit of the graded* R*-ring* $H^\bullet(\Gamma^\bullet)$*.*

(3) Every twisting map of cochain R*-rings* $\sigma^{\bullet,\bullet} : \Gamma^\bullet \otimes \Omega^\bullet \to \Omega^\bullet \otimes \Gamma^\bullet$ *induces a twisting map of graded* R*-rings* $\overline{\sigma}^{\bullet,\bullet} : H^\bullet(\Gamma^\bullet) \otimes H^\bullet(\Omega^\bullet) \to H^\bullet(\Omega^\bullet) \otimes H^\bullet(\Gamma^\bullet)$*.*

In the case of R-corings, the twisting map is defined in a dual manner. Let C and D be two R-corings.

Definition 1.8: A *twisting map of* R*-corings* between C and D is $\tau : C \otimes D \to D \otimes C$, a map that is compatible with the comultiplications of C and D and with counits. More precisely, the following equalities hold true:

$$(\Delta_D \otimes \mathrm{Id}_C) \circ \tau = (\mathrm{Id}_D \otimes \tau) \circ (\tau \otimes \mathrm{Id}_D) \circ (\mathrm{Id}_C \otimes \Delta_D),$$
$$(\mathrm{Id}_D \otimes \Delta_C) \circ \tau = (\tau \otimes \mathrm{Id}_C) \circ (\mathrm{Id}_C \otimes \tau) \circ (\Delta_C \otimes \mathrm{Id}_D),$$
$$(\mathrm{Id}_D \otimes \varepsilon_C) \circ \tau = \varepsilon_C \otimes \mathrm{Id}_D \text{ and } (\varepsilon_D \otimes \mathrm{Id}_C) \circ \tau = \mathrm{Id}_C \otimes \varepsilon_D.$$

As in the case of R-rings, the image of two elements through the twisting map will be denoted by $\tau(c \otimes d) = \sum d_\tau \otimes c_\tau, \forall c \in C, d \in D$.

Moreover, the tensor product $C \otimes D$ has a canonical R-coring structure, which is denoted by $C \otimes_\tau D$. The counit is simply $\varepsilon_C \otimes \varepsilon_D$ and the comultiplication is given by:

$$\Delta = (\mathrm{Id}_C \otimes \tau \otimes \mathrm{Id}_D) \circ (\Delta_C \otimes \Delta_D).$$

Furthermore, in the case of graded corings, the twisting map itself is called *graded* whenever $\tau(C_p \otimes D_q) \subseteq D_q \otimes C_p$, for all p, q. The restriction of τ to $C_p \otimes D_q$ will be denoted by $\tau_{p,q}$ and with these at hand, the twisted tensor product coring $C \otimes_\tau D$ becomes a graded R-coring, whose homogeneous component of degree n is the direct sum of all bimodules $C_p \otimes D_q$ such that $p + q = n$.

We can proceed in a similar way to define and study twisting maps of chain corings. As in the case of cochain rings, a twisting map $\tau_{\bullet,\bullet}$ of such structures is defined such that its components $\tau_{\bullet,p}$ and $\tau_{q,\bullet}$ are morphisms of complexes in the appropriate context.

The dual result with respect to homology is the following.

Proposition 1.3: ([JPS, Proposition 4.6]) *Let* $(\Omega_\bullet, d_\bullet^\Omega)$ *and* $(\Gamma_\bullet, d_\bullet^\Gamma)$ *be two chain corings and* V *an* R*-bimodule.*

(1) If $\varphi_\bullet : (V \otimes \Gamma_\bullet, \mathrm{Id}_V \otimes d_\bullet^\Gamma) \to (\Gamma_\bullet \otimes V, d_\bullet^\Gamma \otimes \mathrm{Id}_V)$ *is a morphism of complexes that is compatible with the counit and the comultiplication of* $\Gamma_\bullet$*, then* $\overline{\varphi}_\bullet = H_\bullet(\varphi_\bullet)$ *is also compatible with the comultiplication and the counit of the graded* R*-coring* $H_\bullet(\Gamma_\bullet)$*.*

(2) If $\chi_\bullet : (\Omega_\bullet \otimes V, d_\bullet^\Omega \otimes \mathrm{Id}_V) \to (V \otimes \Gamma_\bullet, \mathrm{Id}_V \otimes d_\bullet^\Gamma)$ *is a morphism of complexes that is compatible with the counit and the comultiplication of* $\Omega_\bullet$*, then* $\overline{\chi}_\bullet = H_\bullet(\chi_\bullet)$ *is also compatible with the comultiplication and the counit of the graded* R*-coring* $H_\bullet(\Omega_\bullet)$*.*

(3) If $\tau_{\bullet,\bullet} : \Omega_\bullet \otimes \Gamma_\bullet \to \Gamma_\bullet \otimes \Omega_\bullet$ *is a twisting map of chain corings, then* $\tau_{\bullet,\bullet}$ *induces a twisting map of graded* R*-corings:*

$$\overline{\tau}_{\bullet,\bullet} : H_\bullet(\Omega_\bullet) \otimes H_\bullet(\Gamma_\bullet) \to H_\bullet(\Gamma_\bullet) \otimes H_\bullet(\Omega_\bullet).$$

Now the following subject which we briefly recall is that of *entwining maps*, which allow a kind of twisting, but for a tensor product of an R-ring and an R-coring. Let A be an R-ring and C be an R-coring.

Definition 1.9: A bimodule morphism $\lambda : C \otimes A \to A \otimes C$ is called *an entwining map* if it is compatible with the unit of A, the counit of C, the multiplication of A and the comultiplication of C. That is, the following relations hold true:

$$\lambda \circ (\mathrm{Id}_C \otimes m_A) = (m_A \otimes \mathrm{Id}_C) \circ (\mathrm{Id}_A \otimes \lambda) \circ (\lambda \otimes \mathrm{Id}_A),$$
$$(\mathrm{Id}_A \otimes \Delta_C) \circ \lambda = (\lambda \otimes \mathrm{Id}_C) \circ (\mathrm{Id}_C \otimes \lambda) \circ (\Delta_C \otimes \mathrm{Id}_A),$$
$$\lambda(c \otimes 1_A) = 1_A \otimes c \text{ and } (\mathrm{Id}_A \otimes \varepsilon_C) \circ \lambda = \varepsilon_C \otimes \mathrm{Id}_A.$$

If both A and C are graded, the entwining map itself is called *graded* if it respects the gradings. That is, $\lambda(C_p \otimes A^q) \subseteq A^q \otimes C_p$. The restriction of λ to $C_p \otimes A^q$ will be denoted by λ_p^q.

In the case when $(\Omega^\bullet, d_\Omega^\bullet)$ is a cochain R-ring and $(\Gamma_\bullet, d_\bullet^\Gamma)$ is a chain R-coring, a graded entwining map $\lambda_\bullet : \Gamma_\bullet \otimes \Omega^\bullet \to \Omega^\bullet \otimes \Gamma^\bullet$ is called a *differential entwining map* if $\lambda_p^\bullet$ and $\lambda_\bullet^q$ are morphisms of complexes, for all p and q.

Moreover, in this case, a differential entwining map induces a graded entwining map:

$$\overline{\lambda}_\bullet^\bullet : H_\bullet(\Gamma_\bullet) \otimes H^\bullet(\Omega^\bullet) \to H^\bullet(\Omega^\bullet) \otimes H_\bullet(\Gamma_\bullet).$$

Let us now describe how a special twisting map τ' and two precise entwining maps λ and ν are defined, in order to discuss (almost) Koszul pairs.

Let A and B be two strongly graded, connected R-rings. Then we have the almost Koszul pairs $(A, T(A))$ and $(B, T(B))$. For short, let us denote $C = T(A)$ and $D = T(B)$. By definition, we have two R-bimodule isomorphisms: $\theta_{C,A} : C_1 \to A^1$ and the second, $\theta_{D,B} : D_1 \to B^1$. Assume we are given a graded twisting map $\sigma : B \otimes A \to A \otimes B$, that is invertible.

Define $\varphi^{B,\overline{A}} = \sigma^{-1}\mid_{\overline{A}\otimes B}$, we obtain an entwining map $\varphi_\bullet^{B,\overline{A}}$ between $\Omega_\bullet(A)$ and B, which is a morphism of complexes. Now, sine any right R-module is flat (because of the semisimplicity of the base ring R), we have:

$$H_p(\Omega_\bullet(A)\otimes B) \simeq H_p(\Omega_\bullet(A))\otimes B = C_p \otimes B,$$

and also

$$H_p(B\otimes\Omega_\bullet(A)) \simeq B\otimes H_p(\Omega_\bullet(A)) = B\otimes C_p.$$

Therefore, we have an induced morphism $\lambda : C\otimes B \to B\otimes C$, that is a graded entwining map.

By symmetry, taking $\varphi^{\overline{B},A} = \sigma^{-1}\mid_{A\otimes\overline{B}}$, then $\overline{\varphi}_\bullet^{\overline{B},A}$ induces a graded entwining map, which we denote by $\nu : A\otimes D \to D\otimes A$.

Now we can also construct a twisting map between C and D. Take $\psi^{\overline{B},C} = \lambda\mid_{C\otimes\overline{B}}$ and get a twisting map $\overline{\psi}_\bullet : C\otimes\Omega_\bullet(B) \to \Omega_\bullet(B)\otimes C$ of graded corings. Moreover, this is actually a map of chain corings, since C can be seen as a chain coring with a trivial differential. Thus, $\psi_\bullet^{\overline{B},C}$ induces a graded twisting map $\tau' : C\otimes D \to D\otimes C$.

Given all these data, we can state the main result, in a more condensed form than in [JPS].

Theorem 1.4: ([JPS, Proposition 4.17 and Theorem 4.18]) *Keeping the notations and the assumptions above, the pair* $(A\otimes_\sigma B, C\otimes_\tau D)$ *is almost Koszul.*

Moreover, if (A, C) *and* (B, D) *are in fact Koszul pairs and additionally, all of the twisting and entwining maps* $\sigma, \tau', \lambda, \nu$ *constructed above are invertible, then the pair* $(A\otimes_\sigma B, C\otimes_\tau D)$ *is Koszul as well.*

1.4 Braided Bialgebras

Another category of structures which will be used in the thesis is that of braided bialgebras. Following [B] and [JPS, §6], we will recall the definition of braided bialgebras and present the relevant results regarding Koszulity.

Definition 1.10: Let V be an R-bimodule and $\mathfrak{c} : V\otimes V \to V\otimes V$ be an R-bimodule map. The pair $(V, \mathfrak{c})$ is called a *braided* R*-bimodule* if $\mathfrak{c}$ satisfies the *braid equation*:

$$\mathfrak{c}_1\circ\mathfrak{c}_2\circ\mathfrak{c}_1 = \mathfrak{c}_2\circ\mathfrak{c}_1\circ\mathfrak{c}_2,$$

where $\mathfrak{c}_1 = \mathfrak{c}\otimes \mathrm{Id}_V$ and $\mathfrak{c}_2 = \mathrm{Id}_V\otimes\mathfrak{c}$.

A morphism of braided bimodules $f : (V, \mathfrak{c}_V) \to (W, \mathfrak{c}_W)$ is a bimodule map such that $\mathfrak{c}_W\circ(f\otimes f) = (f\otimes f)\circ\mathfrak{c}_V$.

In the special case when A is an R-ring, the quadruple $(A, m, 1, \mathfrak{c})$ is a *braided* R*-ring* if $(A, \mathfrak{c})$ is a braided R-bimodule, $(A, m, 1)$ is an R-ring and $\mathfrak{c}$ is a twisting map of R-rings. A morphism of braided R-rings is a morphism of R-rings which additionally is a morphism of braided bimodules. Such a braided R-ring is called *braided commutative* or $\mathfrak{c}$-commutative if $m\circ\mathfrak{c} = m$.

In the graded case, we will say that $(A, m, 1, \mathfrak{c})$ is a *graded braided* R*-ring* if and only if $\mathfrak{c}$ is a graded twisting map of R-rings. In this case, the restriction of the braiding $\mathfrak{c}$

to $A^p \otimes A^q$ will be denoted by $\mathfrak{c}^{p,q}$. In this case, one can consider the twisted tensor product $A \otimes_{\mathfrak{c}} A$.

The dual notion, that of braided graded R-corings is defined as follows. Recall that a bialgebra is a set which has simultaneously an algebra structure, a coalgebra structure and the algebra multiplication and the unit are coalgebra maps (or, equivalently, the coalgebra comultiplication and the counit are algebra maps). For details, we point the reader to [DNR, §4.1]. Braided bialgebras were introduced by Takeuchi in [Tak]. A more general notion, following [JPS, §6.1] is that of braided R-bialgebras, which consist of a sextuple $(A, m, 1, \Delta, \varepsilon, \mathfrak{c})$ such that $(A, m, 1, \mathfrak{c})$ is a braided R-ring, $(A, \Delta, \varepsilon, \mathfrak{c})$ is a braided R-coring and Δ, ε are morphisms of R-rings. Additionally the R-bimodule $A \otimes A$, one must consider the R-ring structure $A \otimes_{\mathfrak{c}} A$. A braided R-bialgebra is called *graded* if the underlying ring and coring structures are so.

Two particular examples are of interest, namely the free R-ring $T^a_R(V)$ and the symmetric R-ring $S_R(V, \mathfrak{c})$ for a braided R-bimodule $(V, \mathfrak{c})$. It is proved in [AMS] that there exists a unique R-bimodule map:

$$\mathfrak{c}_T : T^a_R(V) \otimes T^a_R(V) \to T^a_R(V) \otimes T^a_R(V),$$

which extends $\mathfrak{c}$ and is a solution of the braid equation that respects the grading on the tensor product $T^a_R(V) \otimes T^a_R(V)$. Because $\mathfrak{c}$ is a solution of the braid equation, $\mathfrak{c}_T$ is a twisting map of R-rings. Therefore, $(T^a_R(V), \mathfrak{c}_T)$ is a graded braided R-ring. From the universal property of $T^a_R(V)$, one can construct a comultiplication map denoted by $\Delta : T^a_R(V) \to T^a_R(V) \otimes_{\mathfrak{c}_T} T^a_R(V)$ that is unique, such that $\Delta(v) = v \otimes 1 + 1 \otimes v$, for any $v \in V$. Moreover, $\varepsilon : T^a_R(V) \to R$ is the unique R-ring morphism that coincides with Id_R on the zero degree component of $T^a_R(V)$ and vanishes on V. With all this data, in [AMS] is proved that $(T^a_R(V), \Delta, \varepsilon, \mathfrak{c}_T)$ is a braided R-bialgebra.

Now, let $(V, \mathfrak{c})$ be a braided bimodule, which is also *symmetric*, that is $\mathfrak{c}^2 = \mathrm{Id}_{V \otimes V}$. Define the space $W = \mathrm{Im}(\mathrm{Id}_{V \otimes V} - \mathfrak{c})$, which contains only primitive elements. Then, the two-sided ideal I generated by W is a coideal in $T^a_R(V)$. Define $S(V, \mathfrak{c}) = T^a_R(V)/I$. This is a braided R-bialgebra and we denote its braiding by $\mathfrak{c}_S$. We call $S_R(V, \mathfrak{c})$ the *braided symmetric* R-ring of $(V, \mathfrak{c})$. Note that by construction, this is $\mathfrak{c}_S$ commutative.

The most important result regarding Koszulity is the following.

Theorem 1.5: ([JPS, Theorem 6.2]) *Let* R *be a separable algebra over a field* $\Bbbk$ *of characteristic zero. Let* $(A, \mathfrak{c}_A)$ *and* $(H, \mathfrak{c}_H)$ *be two connected graded braided* R*-bialgebras such that* $A^1 = H^1$ *and* $\mathfrak{c}^{1,1}_A = -\mathfrak{c}^{1,1}_H$.

If A *and* H *are strongly graded and braided commutative (as* R*-rings), then* (A, H) *and* (H, A) *are Koszul pairs. In particular,* $(S_R(V, \mathfrak{c}), S_R(V, -\mathfrak{c}))$ *is a Koszul pair.*

It follows immediately that the braided symmetric R-ring of a bimodule is always a Koszul R-ring and a Koszul R-coring with respect to the opposite braiding. This remark and the theorem will come in handy when discussing applications in the category of Koszul posets (see Chapter 3).

CHAPTER 2

KOSZUL RINGS AND CORINGS

In this chapter, we will state and prove our main results regarding Koszul rings and corings. Thus, using the preliminary notions and results presented in the previous chapter, we will put together a comprehensive theorem which characterises Koszul rings through seven equivalent properties. Then, we will use this as a starting point and make the natural passing to the dual notion, that of *Koszul corings*. For these new structures, we will explain how the definition sits in the context and also formulate and prove a theorem that characterises them, by way of duality with respect to the case of Koszul rings.

Moreover, in the last part of the chapter, we will use the Ext ring to provide more characterisations and properties for Koszul rings.

Applications and concrete examples are discussed in the subsequent chapter.

All the results in this chapter are original and were obtained in the articles [MS1], [MS2], and [M].

2.1 Characterisation of Koszul Rings

As a follow-up to the previous chapter, we have seen that the tool of Koszul pairs is appropriate for the study of Koszul R-rings. By means of this tool, we will state and prove equivalent characterisations for Koszul R-rings, of which some are new, some are given a new proof in our context. This approach will best suit the next section, where we define Koszul R-corings and see that similar results hold true for them as well.

The main theorem follows.

Theorem 2.1: *Let A be a connected and strongly graded R-ring. The following are equivalent:*

(1) The R-ring A is Koszul.

(2) The pair $(A, T(A))$ is Koszul.

(3) The pair $(A, A^!)$ is Koszul.

(4) The canonical R-coring morphism $\phi^A : A^! \to T(A)$ is an isomorphism.

(5) The R-coring $T(A)$ is strongly graded.

(6) Any primitive element of $\mathsf{T}(A)$ *is homogeneous of degree 1, that is* $\mathsf{PT}(A) = A^1$.

(7) If $n \neq m$*, then* $\mathsf{T}_{n,m}(A) = 0$*, that is* $\mathsf{T}(A)$ *is* diagonal.

Proof: Firstly, we remark that the implications $(2) \Longrightarrow (1)$ and $(3) \Longrightarrow (1)$ are obvious, since, for example, $K^l_\bullet(A, \mathsf{T}(A))$ is a projective resolution of the base ring R that satisfies the conditions from the definition of Koszul R-rings (Definition 1.1).

For $(1) \Longrightarrow (7)$, assume that A is Koszul and fix a resolution $(P_\bullet, d_\bullet)$ of R such that the nth term P_n is graded and generated by $P_{n,n}$, its homogeneous component of degree n. Therefore, $\mathsf{T}_{n,m}(A)$ is the nth homology group of the complex denoted by $K_\bullet(m) = (R \otimes_A P_\bullet)_m$. Computing explicitly, $K_n(m)$ is the quotient of $P_{n,m}$ by the submodule $\sum_{i=1}^m A^i P_{n,m-i}$. Note that the differentials of this complex are induced by those of the initial resolution and since P_n is generated by $P_{n,n}$, it follows that the complex is diagonale, i.e. $K_n(m) = 0$ whenever $n \neq m$, as required.

In order to prove $(7) \Longrightarrow (3)$, we will construct explicitly a Koszul complex for the pair $(A, A^!)$. Note that in general, this pair is almost Koszul, see §1.2.1. Thus, after the construction of the complex, we need only to prove it exactness in degree n. Let us proceed by induction. The sequence:

$$A \otimes A^!_1 \to A \otimes A^!_0 \to R \to 0$$

is clearly exact, since by construction $A^!_1 = A^1$ and $A^!_0 = R$, so the sequence reduces to:

$$A \otimes A^1 \to A \to R \to 0,$$

the leftmost map being induced by the multiplication of A. The strong grading of A implies that the sequence is exact.

Assume, for the induction step, that the following sequence is exact:

$$A \otimes A^!_n \xrightarrow{d^l_n} A \otimes A^!_{n-1} \xrightarrow{d^l_{n-1}} \dots \xrightarrow{d^l_1} A \otimes A^!_0 \xrightarrow{d^l_0} R \to 0. \tag{2.1}$$

We need to prove that the complex $K^l_\bullet(A, A^!)$ is exact in degree n. Denote as usual the module of n-cycles of this complex by Z_n. Therefore, we have the exact sequence:

$$0 \to Z_n \to A \otimes A^!_n \to Z_{n-1} \to 0. \tag{2.2}$$

The homogeneous components of $A \otimes A^!_n$ are zero in degree less than or equal to n and we also know that d_n is injective on $R \otimes A^!_n$ (as $A^!$ is strongly graded), which makes the i-degree component $Z_{n,i}$ of Z_n null, for all $n \geq i$.

We can complete the sequence (2.1) to a projective graded resolution of R in the category of A-modules:

$$\cdots \to P_{n+2} \to P_{n+1} \to A \otimes A^!_n \to A \otimes A^!_{n-1} \to \cdots \to A \otimes A^!_0 \to R \to 0. \tag{2.3}$$

In particular, combining it with the sequence (2.2), we get a resolution of Z_{n-1} by projective graded left A-modules:

$$\dots P_{n+2} \to P_{n+1} \to A \otimes A^!_n \to Z_{n-1} \to 0. \tag{2.4}$$

A comparison of the last two resolutions gives $\mathrm{Tor}^A_{n+1,m}(R, R) \simeq \mathrm{Tor}^A_{1,m}(R, Z_{n-1})$, $\forall m$.

Therefore, using the sequence (2.2), we get that

$$0 \to \mathrm{Tor}^A_{1,m}(R, Z_{n-1}) \to (R \otimes_A Z_n)_m \to (R \otimes A^!_n)_m \to (R \otimes_A Z_{n-1})_m \to 0$$

is an exact sequence for all m. Also note that $R \otimes A^!_n \simeq A^!_n$ contains only homogeneous elements of degree n.

Let us fix $m > n+1$. By hypothesis, $\mathrm{Tor}^A_{n+1,m}(R, R) = 0$, so this implies that $\mathrm{Tor}^A_{1,m}(R, Z_{n-1}) = 0$ and using the sequence above we get further that $(R \otimes_A Z_n)_m = 0$ as well.

By a well-known formula $(A/A_+) \otimes_A Z_n \simeq Z_n/A_+Z_n$, so $Z_{n,m} = (A_+Z_n)_m$. But we already know that $Z_{n,i} = 0$ for $i \leq n$, so:

$$Z_{n,m} = A^{m-n-1}Z_{n,n+1} + \cdots + A^1 Z_{n,m-1}.$$

In particular, $Z_{n,n+2} = A^1 Z_{n,n+1}$ and by the strong grading on A, we know that $A^{n-m-1} = A^1 A^{n-m-2}$ so by induction on $m > n$ it follows that $Z_{n,m} = A^{m-n-1}Z_{n,n+1}$. This implies that Z_n is generated by $Z_{n,n+1}$ and we remark that d^l_{n+1} induces an isomorphism between the component of degree $n+1$ of $K^l_n(A, A^!)$ that coincides with $R \otimes A^!_{n+1}$ and $Z_{n,n+1}$. This concludes the proof of this implication.

For (3) $\Longrightarrow$ (4), we know that there is a morphism between the Koszul and the bar complexes, $\phi_\bullet : K^l_\bullet(A, A^!) \to \beta^l_\bullet(A)$, as explained in §1.2.1 and this map lifts the identity of R. Both of the complexes are resolutions of R by projective left A-modules, so the Comparison Theorem ([We, Theorem 2.2.6, p. 35]), the morphism $\{\phi_n\}_{n\in\mathbb{N}}$ is invertible up to homotopy in the category of complexes of left A-modules. Therefore, $\{\mathrm{Id}_R \otimes_A \phi_n\}_{n\in\mathbb{N}}$ has the same property in the category of complexes of R-modules. Hence, by taking homology, we get the required isomorphism:

$$H_n(\mathrm{Id}_R \otimes_A \phi_\bullet) = \phi^A_n : A^! \to T_n(A),$$

Now assume that the canonical morphism of R-corings $A^! \to T(A)$ is an isomorphism, which makes $T(A)$ strongly graded, since $A^!$ is always so. This proves the implication (4) $\Longrightarrow$ (5).

Using Lemma 1.3(2) and (3) one proves immediately that the implications (5) $\Longrightarrow$ (6) and (6) $\Longrightarrow$ (7) hold true, after remarking that A is strongly graded, which makes $T_{1,m} = 0$, $\forall m \neq 1$ and $T_{0,m} = 0$, $\forall m > 0$.

Now we are left with proving (3) $\Longrightarrow$ (2) and we will use the fact that (3) and (4) are equivalent so the canonical map $A^! \to T(A)$ is an isomorphism of graded R-corings. Therefore, as previously $(\mathrm{Id}_A, \phi_A) : (A, A^!) \to (A, T(A))$ is an isomorphism of almost Koszul pairs. But the construction of the complex $K^l_\bullet(A, C)$ is natural in the pair (A, C) so $K^l_\bullet(A, A^!)$ and $K^l_\bullet(A, T(A))$ are isomorphic. In particular, the latter complex is exact and this concludes the proof. □

Furthermore, we can retrieve a very well-known property of Koszul rings (and algebras), that they are *quadratic*, i.e. can be presented by generators and relations, the latter being of degree two. Our approach will rely on two preliminary results, which in fact express more general properties, that may be of interest in their own right.

Lemma 2.1: *Let A be a connected, strongly graded R-ring, $A = \oplus_{n\geq 0} A^n$. Then the following* *sequence is exact:*

$$0 \to \mathrm{Tor}^{A}_{2,m}(R,R) \to \mathrm{Tor}^{A/A^{\geq m}}_{2,m}(R,R) \to A^m \to 0,$$

where we have denoted by $A^{\geq m}$ *the ideal* $\oplus_{p\geq m} A^p$.

Proof: Since we know that A is strongly graded, for any positive 2-partition $\mathfrak{m}$ of m, the map $A^{\mathfrak{m}} \xrightarrow{d_2^{\mathfrak{m}}} A^m$ is surjective, so the following sequence is exact:

$$0 \to \mathrm{Ker} d_2^m \to \bigoplus_{\mathfrak{m}\in\mathcal{P}_2(m)} A^{\mathfrak{m}} \to A^m \to 0. \tag{2.5}$$

Now, from the subcomplex $\Omega_\bullet(A, m)$ (sequence (1.8)), we get:

$$\mathrm{Tor}^{A}_{2,m}(R,R) = \mathrm{Ker} d_2^m / \mathrm{Im} d_3^m. \tag{2.6}$$

But we could also work with the algebra $A/A^{\geq m}$ and use the corresponding subcomplex $\Omega_\bullet(A/A^{\geq m}, m)$. In degree 1, this complex is trivial and in rest, for any m, we have:

$$\mathrm{Tor}^{A/A^{\geq m}}_{2,m}(R,R) = \bigoplus_{\mathfrak{m}\in\mathcal{P}_2(m)} A^{\mathfrak{m}} / \mathrm{Im} d_3^m. \tag{2.7}$$

We can now put all these equations and sequences together and obtain readily:

$$0 \to \mathrm{Ker} d_2^m / \mathrm{Im} d_3^m \to \bigoplus_{\mathfrak{m}\in\mathcal{P}_2(m)} A^{\mathfrak{m}} / \mathrm{Im} d_3^m \to A^m \to 0,$$

which is exact, due to the exactness of the sequence (2.5) and the two relations the followed it help obtain the required sequence. □

The second preliminary result, also formulated in a more general setting is the following:

Proposition 2.1: *Let* $\pi : B \to A$ *be a morphism of strongly graded connected* R*-rings. If* m *is a positive integer such that all the components* π^i *are bijective, for all* $0 \leq i < m$ *and also* $\mathrm{Tor}^{A}_{2,m}(R,R) = 0$, *then* π^m *is bijective as well.*

Proof: Using the sequences (2.6) and (2.7) above, it follows that π induces the maps:

$$\widetilde{\pi}^m : \mathrm{Tor}^{B/B^{\geq m}}_{2,m}(R,R) \to \mathrm{Tor}^{A/A^{\geq m}}_{2,m}(R,R)_m \quad \text{and} \quad \overline{\pi}^m : \mathrm{Tor}^{B}_{2,m}(R,R) \to \mathrm{Tor}^{A}_{2,m}(R,R).$$

By construction and definition, the squares of the diagram below are commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Tor}^{B}_{2,m}(R,R) & \xrightarrow{i_B^m} & \mathrm{Tor}^{B/B^{\geq m}}_{2,m}(R,R) & \xrightarrow{p_B^m} & B^m & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \overline{\pi}^m} & & \downarrow{\scriptstyle \widetilde{\pi}^m} & & \downarrow{\scriptstyle \pi^m} & & \\
0 & \longrightarrow & \mathrm{Tor}^{A}_{2,m}(R,R) & \xrightarrow[i_A^m]{} & \mathrm{Tor}^{A/A^{\geq m}}_{2,m}(R,R) & \xrightarrow[p_A^m]{} & A^m & \longrightarrow & 0
\end{array}$$

The homogeneous components of the graded R-rings $A/A^{\geq m}$ and $B/B^{\geq m}$ are zero in degree $i \geq m$ and since π^i is bijective for all $i < m$ by assumption, we deduce that

π induces an isomorphism at the level of normalised complexes, between $\Omega_\bullet(B/B^{\geq m})$ and $\Omega_\bullet(A/A^{\geq})$. In particular, $\widetilde{\pi}^m$ is an isomorphism of R-bimodules. Now, using the Snake Lemma, $\mathrm{Ker}\pi^m \simeq \mathrm{Coker}\overline{\pi}^m$ and since $\mathrm{Tor}^A_{2,m}(R,R) = 0$, we get that the latter is zero, which makes the former null as well. □

Now we can characterise quadraticity in our context and we will make use of the notations from the first section where we introduced the shriek structures. As such, to a connected graded R-ring A, one can associate the R-ring $\langle A^1, K_A\rangle$, where we denote $K_A = \mathrm{Ker}(\mu^{1,1} : A^1 \otimes A^1 \to A^2)$. There exists a unique R-ring morphism $\phi_A : \langle A^1, K_A\rangle \to A$ which lifts $\mu^{1,1}$, by the universal property of the tensor algebra. This morphism is surjective, provided that A is strongly graded. Following the lines of [BGS, Definition 1.2.2], we say that A is *quadratic* if and only if A is connected and strongly graded and ϕ_A is an isomorphism.

Let A now be a fixed connected strongly graded R-ring and denote by $V = A^1$. We make the preparations necessary to characterise quadraticity of an R-ring using the notions developed thus far.

The kernel of the canonical R-ring map from the tensor algebra of the R-bimodule V to A will be denoted by $\widetilde{K}_A = \sum_{m\in\mathbb{N}} \widetilde{K}^m_A$. It is clear that the homogeneous component of degree m of $\widetilde{K}_A$ coincides with the kernel of the iterated multiplication map that we denote by $\mu(m) : V^{(m)} \to A^m$ and moreover that $\widetilde{K}^m_A \subseteq \langle K_A\rangle^m$.

Denote as usual by $Z_{2,m} = \mathrm{Ker} d^m_2$ and $B_{2,m} = \mathrm{Im} d^m_3$. The 2-cycle condition for $\{a_{\mathfrak{m}}\}_{\mathfrak{m}\in\mathcal{P}_2(m)}$ is that every $a_{\mathfrak{m}} \in A^{\mathfrak{m}}$ and $\sum_{\mathfrak{m}\in\mathcal{P}_2(m)} \mu^{\mathfrak{m}}(a_{\mathfrak{m}}) = 0$. Also, for any positive integer m, denote by $\alpha_m(x) = \{\mu(\mathfrak{m})(x)\}_{\mathfrak{m}\in\mathcal{P}_2(m)}$.

With this notation, it is easy to see that $\alpha_m(x) \in Z_{2,m}$, since x is an element in $\mathrm{Ker}\mu(m) = \mathrm{Ker}\mu^{\mathfrak{m}} \circ \mu(\mathfrak{m})$, for all $\mathfrak{m} \in \mathcal{P}_2(m)$. Therefore, the correspondence $x \mapsto \alpha(x)$ defines an R-bimodule map $\alpha_m : \langle K_A\rangle^m \to Z_{2,m}$.

Moreover, $\forall x \in \widetilde{K}^m_A \Rightarrow \alpha_m(x) \in B_{2,m}$. That happens because by definition of $\widetilde{K}^m_A$, it is enough to show that $\alpha_m(x)$ is a boundary for all elements $x \in V^{(i-1)} \otimes K_A \otimes V^{(m-i-1)}$ and $1 \leq i \leq m-1$. So let $i = 1$ and since $x \in K_A \otimes V^{(m-2)}$, it follows that we can write $\alpha_m(x) = \{x_{\mathfrak{m}}\}_{\mathfrak{m}\in\mathcal{P}_2(m)}$, where

$$x_{\mathfrak{m}} = \begin{cases} (\mathrm{Id}_V \otimes \mu(m-1))(x), & \text{if } \mathfrak{m} = (1, m-1); \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $\alpha_m(x) = -d^m_3(y)$, where $y = (V^{(2)} \otimes \mu(m-2))(x)$. For all other values of i, the proof for the fact that $\alpha_m(x) \in B_{2,m}$ is similar.

In conclusion, α_m defines a map $\widetilde{K}^m_A \to B_{2,m}$, which we denote by abuse still with α_m. In other words, the diagram below is commutative, where the vertical arrows are the canonical inclusions.

$$\begin{array}{ccc} \widetilde{K}^m_A & \xrightarrow{\alpha_m} & B_{2,m} \\ \downarrow & & \downarrow \\ \langle K_A\rangle^m & \xrightarrow{\alpha_m} & Z_{2,m} \end{array}$$

Now a last lemma which will enable us to state the quadraticity in a slightly different way.

Lemma 2.2: *Let* A *be a connected strongly graded* R*-ring. Keeping the notations and context used above, if* $\widetilde{K}_A^m = \langle K_A \rangle^m$*, then* $\mathrm{Tor}^A_{2,m}(R, R) = 0$.

Proof: Let $\xi = a_1 \cdots a_i \otimes a_{i+1} \cdots a_m$, where $a_1, \ldots, a_m$ are arbitrary in $V = A^1$. From:

$$\xi = a_1 \cdots a_{i-1} \otimes a_i \cdots a_m + d_3^m(a_1 \cdots a_{i-1} \otimes a_i \otimes a_{i+1} \cdots a_m),$$

we can deduce inductively that $\xi = a_1 \otimes a_2 \cdots a_m + d_3^m(\omega)$ for some $\omega \in \Omega_3(A, m)$.

Let $x \in Z_{2,m}$. From the strong grading of A, using the remarks that preceded this lemma, there must exist $y \in A^1 \otimes A^{m-1}$ and $z \in \Omega_3(A, m)$ such that $x = y + d_3^m(z)$. Also, y is a 2-cycle.

The purpose is to prove that x is a boundary and for this it suffices to show that $y \in B_{2,m}$. From the strong grading on A we deduce that the map $f = \mathrm{Id}_V \otimes \mu(m-1)$ is surjective. Therefore, there must exist $y' \in V^{(m)}$ such that $f(y') = y$. Since y is a cycle, we also obtain that $y' \in \langle K_A \rangle^m$, which is also $\mathrm{Ker}\mu(m)$.

By the hypothesis, $\langle K_A \rangle^m = \widetilde{K}_A^m$, so $y' = \sum_{i=1}^{m-1} y_i'$, the components being elements $y_i' \in V^{(i-1)} \otimes K_A \otimes V^{(m-i-1)}$. Since $f(y_i') = 0$, $\forall i > 1$, it follows that $y = f(y_1')$. Also, as $\mu(\mathfrak{m})(y_1') = 0$, for any 2-partition $\mathfrak{m} \in (\mathcal{P}_2(m) - \{(1, m-1)\})$, it follows that $y = \alpha_m(y_1')$. The proof finishes by concluding that y is a boundary by remarking that α_m maps $\widetilde{K}_A^m$ to $B_{2,m}$. □

Now for the awaited result.

Proposition 2.2: *Let* A *be a strongly graded* R*-ring. Then* A *is a quadratic* R*-ring if and only if* $\mathrm{Tor}^A_{2,m}(R, R) = 0$, $\forall m \geq 3$.

Proof: Let $B = \langle A, K_A \rangle$, employing the notations used previously. Assume first that $T_{2,m}(A) = \mathrm{Tor}^A_{2,m}(R, R) = 0$, for all $m \geq 3$. It is enough to prove that the components ϕ_A^m of the canonical map $\phi_A : B \to A$ are all isomorphisms, since B is obviously quadratic, by construction. We proceed by induction on m. The maps ϕ_A^0 and ϕ_A^1 are injective, because they coincide with the identity maps of R and A^1, respectively. Assume now that ϕ_A^k is bijective for all $k \leq m-1$ and $m \geq 3$. Then, since $T_{2,m}(A) = 0$, $\forall m \geq 3$, we get, using Proposition 2.1 that ϕ_A^m is also bijective.

For the converse, one can use directly the preceding lemma, since $\widetilde{K}_A^m = \langle K_A \rangle^m$, for all m, by the definition of quadratic R-rings, so B coincides with A. □

2.2 Koszul Corings - Definitions and Properties

Once Koszul pairs are introduced and used in several essential purposes and moreover, once the connection with Koszul rings is made clear (cf. Theorem 1.3), it is natural to try to know more about the second component of a Koszul pair. Furthermore, we aimed at obtaining (at least) the equivalent characterisations for this second component as in the case of Koszul rings (cf. Theorem 2.1). Therefore, in this section, we will proceed with introducing *Koszul corings* as a natural dualisation of Koszul rings. After clearing the basic definitions and notions that are needed to properly introduce this structure, we will pass to formulating and proving a characterisation theorem which corresponds to the case of Koszul rings.

The interplay between both of these structures will show up in the subsequent chapter, where we discuss several applications. In particular, when discussing about graded linear duality, we will see how a Koszul ring can be turned into a Koszul coring and vice versa.

But let us start with the beginning. Let (X, ρ^X) be a graded right comodule over a graded R-coring C (see §1.1.2 for the basic definitions). For every integer n, the map ρ^X defines a morphism of graded C-comodules $X \to X_n \otimes C$, whose component of degree k is $\rho^X_{n,k-n}$, for every k. The component of degree k of $X_n \otimes C$ is $X_n \otimes C_{k-n}$, where by the positive grading on C, we have $C_i = 0$, $\forall i < 0$. Also due to the positive grading, in particular it follows that $\rho^X_{n,k-n} = 0$, $\forall k < n$.

Recall that for a graded right R-module M, one calls M *strongly graded* if the multiplication maps $m_i : M_i \otimes R \to M$ are surjective for all i. In the dual case of graded right C-comodules, we introduce the following:

Definition 2.1: Let $X = \oplus_{n \geq 0} X_n$ be a graded right C-comodule, with structure map ρ^X. We say that X is *strongly graded in degree* n if the morphism of graded C-comodules $\rho^X_n : X \to X_n \otimes C$ is injective.

We remark that, by definition, a graded right C-comodule is strongly graded in degree n if and only if $X_p = 0$, for all $p < n$ and $\rho^X_{n,q} : X_{n+q} \to X_n \otimes C_q$ is injective for all $q \geq 0$.

Some general properties of graded comodules are contained in the following lemma and will be used later on.

Lemma 2.3: *Let C be a strongly graded R-coring and (X, ρ^X) be a graded right C-comodule.*

(1) The inclusion $\mathrm{Ker}\rho^X_{i,m-i} \subseteq \mathrm{Ker}\rho^X_{j,m-j}$ holds for all $0 \leq j \leq i < m$.

(2) There is a canonical isomorphism $\mathrm{Hom}^C(R, X)_m \simeq \bigcap_{i=0}^{m-i} \mathrm{Ker}\rho^X_{i,m-i} = \mathrm{Ker}\rho^X_{m-1,1}$.

(3) X is strongly graded in degree n if and only if $X_i = 0$, for all $i \leq n$ and $\rho^X_{i,1}$ is injective for all $i \geq n$.

(4) Let $f : X \to Y$ be a morphism of graded right C-comodules. If X is strongly graded in degree n and $f_n : X_n \to Y_n$ is an injective morphism, then f is injective as well.

Proof: (1) Consider the following diagram:

$$\begin{array}{ccc}
X_m & \xrightarrow{\quad \rho^X_{i,m-i} \quad} & X_i \otimes C_{m-i} \\
\Big\downarrow{\scriptstyle \rho^X_{j,m-j}} & & \Big\downarrow{\scriptstyle \rho^X_{j,i-j}} \\
X_j \otimes C_{m-j} & \xrightarrow[\mathrm{Id}_{X_j} \otimes \Delta^C_{i-j,m-i}]{} & X_j \otimes C_{i-j} \otimes C_{m-i}
\end{array}$$

Let x be an element of $\mathrm{Ker}\rho^X_{i,m-i}$. Since we know that $\mathrm{Id}_{X_j} \otimes \Delta^C_{i-j,m-i}$ is an injective map, because C is strongly graded, it follows that $\rho^X_{j,m-j}(x) = 0$, therefore $x \in \mathrm{Ker}\rho^X_{j,m-j}$, as required.

(2) Let X and Y be two graded C-comodules. Using the definition, $f \in \mathrm{Hom}^C(X, Y)_m$ if and only if f is a morphism of graded C-comodules of degree m, i.e. $f(X_k) \subseteq Y_{m+k}$, for all k. In particular, if $f : R \to X$ is a C-colinear map of degree m, then $f(1) \in X_m$ and $\rho^X(f(1)) = f(1) \otimes 1$. But the former is equivalent to $\rho^X_{i,m-i}(f(1)) = 0$, $\forall 0 \leq i \leq m-1$. We conclude that the required isomorphism is given by the mapping $f \mapsto f(1)$ and the equation from the statement follows by the first part of the lemma.

(3) Given the remark just above the definition of strongly graded C-comodules, the condition $X_i = 0$ for $i < n$ is actually part of the hypothesis for both of the implications.

We must prove that $\rho^X_{i,1}$ is injective for all $i \geq n$ if and only if $\rho^X_{n,q} : X_{n+q} \to X_n \otimes C_q$ is injective for all non-negative integers q. For this, let us proceed by induction on q. The map $\rho^X_{n,0}$ coincides with the canonical isomorphism $X_n \simeq X_n \otimes R$ and $\rho^X_{n,1}$ is injective by assumption. Assume that $\rho^X_{n,q}$ is injective, so that the horizontal arrow on the bottom of the commutative diagram below is injective:

$$\begin{array}{ccc} X_{n+q+1} & \xrightarrow{\rho^X_{n,q+1}} & X_n \otimes C_{q+1} \\ \downarrow{\scriptstyle \rho^X_{n+q,1}} & & \downarrow{\scriptstyle \mathrm{Id}_{X_n} \otimes \Delta^C_{q,1}} \\ X_{n+q} \otimes C_1 & \xrightarrow{\rho^X_{n,q} \otimes \mathrm{Id}_{C_1}} & X_n \otimes C_q \otimes C_1 \end{array}$$

By hypothesis, the leftmost vertical map is injective so it follows that $\rho^X_{n,q+1}$ is also injective.

For the converse, starting from the commutative diagram:

$$\begin{array}{ccc} X_{i+1} & \xrightarrow{\rho^X_{i,1}} & X_i \otimes C_1 \\ \downarrow{\scriptstyle \rho^X_{n,i+1-n}} & & \downarrow{\scriptstyle \rho^X_{n,i-n} \otimes \mathrm{Id}_{C_1}} \\ X_n \otimes C_{i+1-n} & \xrightarrow{\mathrm{Id}_{X_n} \otimes \Delta^C_{i-n,1}} & X_n \otimes C_{i-n} \otimes C_1 \end{array}$$

we note that the component $\rho^X_{n,i+1-n}$ is injective by assumption, and also $\mathrm{Id}_{X_n} \otimes \Delta^C_{i-n,1}$ is as well, from the strong grading on C. The conclusion is that $\rho^X_{i,1}$ is injective, as required.

(4) For this last statement, we have to prove that all components f_k are injective. When $k < n$, the property holds trivially, because by the preceding part of the lemma, $X_k = 0$. Also, the map f_n is injective by assumption, so we can work for $k > n$. The following diagram is commutative, since f is a morphism of graded comodules:

$$\begin{array}{ccc} X_k & \xrightarrow{f_k} & Y_k \\ \downarrow{\scriptstyle \rho^X_{n,k-n}} & & \downarrow{\scriptstyle \rho^Y_{n,k-n}} \\ X_n \otimes C_{k-n} & \xrightarrow{f_n \otimes \mathrm{Id}_{C_{k-n}}} & Y_n \otimes C_{k-n} \end{array}$$

We know that $f_n \otimes \mathrm{Id}_{C_{k-n}}$ is injective by hypothesis. On the other hand, $\rho^X_{n,k-n}$ is also injective, given the strong grading of X in degree n. The conclusion is that $x = 0$ and this completes the proof. □

Now for the main definition of this chapter. We will introduce Koszul corings by dualising [BGS, Definition 1.2.1] and then we will formulate seven characterisations of this class of graded corings in a following theorem. The reader should compare the properties that this theorem formulates with those in Theorem 2.1.

Definition 2.2: A graded connected R-coring C is called *Koszul* if the base ring R has a resolution $0 \to R \to Q^\bullet$ by injective graded right C-comodules such that the nth term Q^n is strongly graded in degree n, for all n. Equivalently, in this case we say Q^n is *cogenerated* in degree n.

The following theorem gives several equivalent characterisations of this class.

Theorem 2.2: *Let* C *be a connected strongly graded* R*-coring. The following are equivalent:*

(1) C *is a Koszul coring.*

(2) The pair $(E(C), C)$ *is Koszul.*

(3) The pair $(C^!, C)$ *is Koszul.*

(4) The canonical morphism $E(C) \to C^!$ *is bijective.*

(5) The R*-ring* $E(C)$ *is strongly graded.*

(6) The R*-bimodule of indecomposable elements (cf. Definition 1.2) of* $E(C)$ *is concentrated in degree 1, that* $QE(C) \simeq E^1(C)$.

(7) The R*-ring* $E(C)$ *is diagonal, i.e.* $E^{n,m}(C) = 0$, $\forall n \neq m$.

Proof: The reader is invited to compare the results that the present theorem states with those for Koszul rings (see Theorem 2.1). A close connection is obvious and this is well expected so it will be no surprise the fact that the proof follows the lines of the aforementioned result, but in a dual way. Let us get on to it.

For any Koszul pair (A, C), we know that the complex $K_r^\bullet(A, C)$ is a resolution for the base ring R which satisfies the conditions from the definition of Koszul corings (see Corollary 1.1). Therefore, we the proofs for the implications $(2) \Longrightarrow (1)$ and $(3) \Longrightarrow (1)$ are done.

Now, to prove $(1) \Longrightarrow (7)$, assume that C is Koszul. By definition, this is equivalent to the fact that R has an injective resolution $0 \to R \to Q^\bullet$ such that each term Q^n is strongly graded in degree n. Recall that $E^{n,m}(C) = \mathrm{Ext}^C_{n,m}(R, R)$ is the nth cohomology group of the complex that is obtained after applying the functor $\mathrm{Hom}^C(R, -)_m$ to this resolution. Hence, it is enough to prove that $\mathrm{Hom}^C(R, Q^n)_m$ is null whenever $m \neq n$. For this, using Lemma 2.3(2), we have that $\mathrm{Hom}^C(R, Q^n)_m = \mathrm{Ker}\rho^{Q^n}_{m-1,1}$. Now the claim follows from the fact that Q^n is strongly graded in degree n, since the same lemma shows that $Q^{n,m} = 0$, whenever $m < n$ and the component $\rho^{Q^n}_{m-1,1}$ is injective for all $m > n$.

For $(7) \Longrightarrow (3)$, assume that $E(C)$ is diagonal. We will prove inductively (on n) that the complex $K_r^\bullet(C^!, C)$ is exact in degree n. To start off, consider the sequence:

$$0 \to R \to C^!_0 \otimes C \to C^!_1 \otimes C.$$

This is exact by the definition of the complex $K^\bullet(C^!, C)$ and using the strong grading on C, [JPS, Lemma 2.1] (recalled as Lemma 1.5) gives the required result.

Assume now that the complex $K^\bullet(C^!, C)$ is exact in all degrees $0 \leq i \leq n-1$. We want exactness in degree n as well. Consider the exact sequence:

$$0 \to R \to C^!_0 \otimes C \xrightarrow{d_r^0} C^!_1 \otimes C \xrightarrow{d_r^1} \dots \xrightarrow{d_r^{n-2}} C^!_{n-1} \otimes C \xrightarrow{d_r^{n-1}} C^!_n \otimes C \xrightarrow{\alpha} Y \to 0, \quad (2.8)$$

where we have denoted by $Y = \mathrm{Coker} d_r^{n-1}$ and α is the canonical projection. The claim is that Y is strongly graded in degree $n+1$. Because $d_r^{n-1}(\widehat{x} \otimes c) = \widehat{x \otimes c} \otimes 1$, for any $x \in C_1^{n-1}$ and $c \in C_1$, it follows that $(C^!_n \otimes C)_n \subseteq \mathrm{Im} d_r^{n-1}$. Therefore, Y has no non-zero elements of degree n. But if Y_k is the homogeneous component of degree k for Y, then $Y_k = 0$ whenever $k < n$, since Y is the quotient of $C^!_n \otimes C$ by a graded subcomodule. Therefore, we are left with proving only the fact that $\rho^Y_{m-1,1}$ is injective for all $m > n+1$.

Denote by $X = \mathrm{Ker}\alpha = \mathrm{Im} d^{n-1}_r$. Starting from the sequence $0 \to X \to C^!_n \otimes C$, we can complete it to a resolution:

$$0 \to X \to C^!_n \otimes C \to Q^{n+1} \to Q^{n+2} \to \dots,$$

using injective graded right C-comodules, and taking it in combination with the sequence (2.8) we obtain a new injective resolution:

$$0 \to R \to C^!_0 \otimes C \to C^!_1 \otimes C \to \cdots \to C^!_n \otimes C \to Q^{n+1} \to Q^{n+2} \to \dots. \tag{2.9}$$

Now we can compute $\mathrm{Ext}^{n+1,m}_C(R,R)$ as the cohomology group in degree $n+1$ for the complex obtained from the above:

$$0 \to \mathrm{Hom}^C(R, C^!_0 \otimes C)_m \to \mathrm{Hom}^C(R, C^!_1 \otimes C)_m \to \cdots$$
$$\cdots \to \mathrm{Hom}^C(R, C^!_n \otimes C)_m \to \mathrm{Hom}^C(R, Q^{n+1})_m \to \cdots .$$

Also, $\mathrm{Ext}^{1,m}_C(R,R)$ is computed as the 1st cohomology group of the complex:

$$0 \to \mathrm{Hom}^C(R, C^!_n \otimes C)_m \to \mathrm{Hom}^C(R, Q^{n+1})_m \to \mathrm{Hom}^C(R, Q^{n+2})_m \to \dots,$$

so we deduce that we have an isomorphism $\mathrm{Ext}^{1,m}_C(R,X) \simeq \mathrm{Ext}^{n+1,m}_C(R,R)$. Since by definition $X = \mathrm{Ker}\alpha$, the long exact sequence that connects the functors $\mathrm{Ext}^{\bullet,m}_C(R,-)_m$ can be written as:

$$0 \to \mathrm{Hom}^C(R,X)_m \to \mathrm{Hom}^C(R, C^!_n \otimes C)_m \to \mathrm{Hom}^C(R,Y)_m \to \mathrm{Ext}^{1,m}_C(R,X) \to \dots.$$

The hypothesis is that $E(C)$ is diagonal, so using the isomorphism that we obtained between the Ext-groups it follows that $\mathrm{Ext}^{1,m}_C(R,X) = 0$, whenever $m > n+1$. Moreover, $\mathrm{Hom}^C(R, C^!_n \otimes C) \simeq \mathrm{Hom}_R(R, C^!_n)$, an isomorphism of graded R-modules, the latter being concentrated in degree n. Therefore, we get that $\mathrm{Hom}^C(R,Y)_m = 0$, for all $m > n+1$ and by Lemma 2.3(2) it follows that $\rho^Y_{m-1,1}$ is injective for all $m > n+1$. Since we know already that $Y_k = 0$ for $k < n+1$, Lemma 2.3(3) enables us to conclude that Y is strongly graded in degree $n+1$, as required.

We now come back to proving exactness of $K^\bullet(C^!, C)$ in degree n and for this we need to show that $\mathrm{Ker}(\partial : Y \to C^!_{n+1})$ is trivial, the map being induced by the differential d^n_r. Using Lemma 2.3(4) plus the fact that Y is strongly graded in degree $n+1$, it is enough to prove that one component of the map is injective, namely $\partial_{n+1} : Y_{n+1} \to C_{n+1}$.

Let $y \in \mathrm{Ker}\partial_{n+1}$. There exist $y_1, \dots, y_p \in C^{(n)}$ and $c_1, \dots, c_p \in C_1$ such that y is the equivalence class of the element $\sum_{i=1}^p \widehat{y}_i \otimes c_i$ in $Y_{n+1} = (C^!_n \otimes C_1)/d^{n-1}_r(C^!_{n-1} \otimes C_2)$. We have denoted by $\widehat{y}_i$ the equivalence class of the element y_i in $C^!_n = C^{(n)}_1/W_n$, where $W_n = \sum_{k=1}^{n-1} C^{(k-1)}_1 \otimes \mathrm{Im}\Delta_{1,1} \otimes C^{(n-k-1)}_1$. Now, we know that $\partial_{n+1}(y) = 0$ and since the differential d^n_r is induced by the multiplication of the shriek ring $C^!$, it follows that the element $y' = \sum_{i=1}^p y_i \otimes c_i$ belongs to the submodule $W_{n+1} \subseteq C^{(n+1)}$, from the construction of $C^!_{n+1}$. Also note that $W_2 = \mathrm{Im}\Delta_{1,1}$. Therefore, one can rewrite the element $y' = x + \sum_{j=1}^{q} y''_j \otimes c^j_{1,1} \otimes c^j_{2,1}$, for some $x \in W_n \otimes C_1$, $y''_1, \ldots, y''_q \in C_1^{(n-1)}$ and $c^1, \ldots, c^q \in C_2$. By computing:

$$\sum_{i=1}^{p} \widehat{y}_i \otimes c_i = \sum_{j=1}^{p} \widehat{y''_j \otimes c^j_{1,1}} \otimes c^j_{2,1} = d^n_r\Big(\sum_{j=1}^{p} \widehat{y''_j \otimes c^j}\Big),$$

we deduce that $y = 0$, which completes the proof of this implication.

For (3) $\Longrightarrow$ (4), we use the morphism of complexes $\phi^\bullet : \beta^\bullet_r(C) \to K^\bullet_r(C^!, C)$ (that we introduced in §1.2.1) which lifts the identity of R. Both of these complexes are resolutions of R by injective right C-comodules, so the Comparison Theorem tells us that the morphism $\{\phi^n\}_{n\in\mathbb{N}}$ is invertible up to a homotopy in the category of complexes of right C-comodules. Hence, $\mathrm{Hom}^C(R, \phi^\bullet)$ has the same property in the category of complexes of right R-modules. Taking the cohomology, we deduce that $H^n(\mathrm{Hom}^C(R, \phi^\bullet))$ is an isomorphism for all $n \geq 0$. Therefore, $\phi^n_C : E^n(C) \to C^!_n$ is an isomorphism, which coincides with $H^n(\mathrm{Hom}^C(R, \phi^\bullet))$.

The implication (4) $\Longrightarrow$ (5) is immediate, because $C^!$ is always strongly graded, so $E(C)$ has the same property. Moreover, (5) $\Longleftrightarrow$ (6) and (5) $\Longrightarrow$ (7) are straightforward from Lemma 1.1.

For (3) $\Longrightarrow$ (2), we remark that there is an isomorphism of complexes between $K^\bullet_r(C^!, C)$ and $K^\bullet_r(E(C))$, which makes the latter exact and this completes the entire proof. □

In what concerns terminology, we could call $K_\bullet(A, A^!)$ the *Koszul complex* of A and, likewise, $K^\bullet(C^!, C)$ to be the *Koszul complex* for C. Indeed, as outlined in the proof of [JPS, Theorem 2.14] (recalled here as Theorem 1.3), the complex $K^l_\bullet(A, A^!)$ is isomorphic to the Koszul complex that was introduced in [BGS, p. 483] and by analogy and duality, one could make the reference to the Koszul complex for corings as well.

Now let us prove another result for Koszul corings which is in close relation to the case of Koszul rings about quadraticity. The main result will be that any Koszul coring is quadratic, but it will come as a consequence to a build up of preliminary results, some of which are more general and useful in their own right. The reader is invited to compare the results that will follow with those in the previous section on Koszul rings.

Lemma 2.4: *Let C be a strongly graded R-coring. The following sequence is exact:*

$$0 \to C_m \to \mathrm{Ext}^{2,m}_{C_{<m}}(R, R) \to \mathrm{Ext}^{2,m}_{C}(R, R) \to 0.$$

Proof: Let us start by remarking that the differential d^1_m of the subcomplex $\Omega^\bullet(C, m)$ maps an element $c \in C_m$ to the family of morphisms $\{\Delta_{\mathfrak{m}}(c)\}_{\mathfrak{m}\in\mathcal{P}_2(m)}$, where we have used the notation $\Delta_{\mathfrak{m}}$ for the component $\Delta_{m_1,m_2} : C_{m_1+m_2} \to C_{m_1} \otimes C_{m_2}$, for any positive 2-partition $\mathfrak{m} = (m_1, m_2)$ of m. We know by hypothesis that $\Delta_{\mathfrak{m}}$ is injective, given the strong grading on C, so d^1_m is injective as well. Therefore, we can form an exact sequence:

$$0 \to C_m \xrightarrow{d^1_m} \mathrm{Ker}d^2_m \to \mathrm{Ker}d^2_m/\mathrm{Im}d^1_m \to 0.$$

Moreover, we know that $\mathrm{Ext}^{2,m}_C(R, R) = \mathrm{Ker}d^2_m/\mathrm{Im}d^1_m$. Also, the complex $\Omega^\bullet(C_{<m}, m)$

coincides in the first degrees with:

$$0 \to 0 \xrightarrow{d^0_m} 0 \xrightarrow{d^1_m} \bigoplus_{m_2 \in \mathcal{P}_2(m)} C_{m_2} \xrightarrow{d^2_m} \bigoplus_{m_3 \in \mathcal{P}_3(m)} C_{m_3}.$$

Therefore, we can conclude the proof with remarking that there is an isomorphism $\mathrm{Ext}^{2,m}_{C_{<m}}(R,R) \simeq \mathrm{Ker} d^2_m$. □

Proposition 2.3: *Let* $\pi : C \to D$ *be a morphism of strongly graded and connected* R*-corings. If* $m \geq 0$ *such that* $\mathrm{Ext}^{2,m}_C(R,R) = 0$ *and all the components* π_i *are bijective for* $0 \leq i < m$*, then* π_m *is bijective as well.*

Proof: As in the dual case (see Proposition 2.1), π induces two morphisms:

$$\overline{\pi}^m : \mathrm{Ext}^{2,m}_C(R,R) \to \mathrm{Ext}^{2,m}_D(R,R) \quad \text{and} \quad \widetilde{\pi}^m : \mathrm{Ext}^{2,m}_{C_{<m}}(R,R) \to \mathrm{Ext}^{2,m}_{D_{<m}}(R,R),$$

which make the following diagram commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_m & \longrightarrow & \mathrm{Ext}^{2,m}_{C_{<m}}(R,R) & \longrightarrow & \mathrm{Ext}^{2,m}_C(R,R) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_m} & & \downarrow{\scriptstyle \widetilde{\pi}_m} & & \downarrow{\scriptstyle \overline{\pi}_m} & & \\ 0 & \longrightarrow & D_m & \longrightarrow & \mathrm{Ext}^{2,m}_{D_{<m}}(R,R) & \longrightarrow & \mathrm{Ext}^{2,m}_D(R,R) & \longrightarrow & 0 \end{array}$$

Therefore, the map $\widetilde{\pi}_m$ induces an isomorphism, because π_i is bijective for all degrees $i < m$. Thus, by the Snake Lemma, we have that $\mathrm{Ker}\pi_m = 0$ and also that $\mathrm{Coker}\pi_m \simeq \mathrm{Ker}\overline{\pi}_m = 0$, which complete the proof. □

In order to introduce the notion of *quadratic corings*, recall that for a connected R-coring C, the family $\{\Delta(n)\}_{n\geq 0}$ of iterated comultiplications $\Delta(n) : C_n \to C_1^{(n)}$ defines a morphism of connected corings $\phi_C : C \to T^c_R(C_1)$. The image of ϕ_C is a subcoring $\overline{C}$ of $\widetilde{C} = \{C_1, \mathrm{Im}\Delta_{1,1}\}$. Therefore, we can see ϕ_C as a coring morphism $C \to \widetilde{C}$ and say that the coring C is *quadratic* whenever this morphism is a bijection. Therefore, C is quadratic if and only if it is strongly graded and $\overline{C} = \widetilde{C}$.

Now let C be a strongly graded R-coring. Proceeding as in the case of R-rings, one can relate $Z^{2,m} = \mathrm{Ker} d^2_m$, which is the set of 2-cocycles in the subcomplex $\Omega^\bullet(C,m)$ to $\widetilde{C}_m$. In order to obtain this, notice that for any 2-cocycle $x = \{x_{\mathfrak{m}}\}_{\mathfrak{m}\in\mathcal{P}_2(m)}$, the element $\Delta(\mathfrak{m})(x_{\mathfrak{m}})$ is independent on $\mathfrak{m} \in \mathcal{P}_2(m)$, since the components of the (iterated) comultiplication are injective, because of the strong grading on C. Recall that by definition of the iterated comultiplication, we have $\Delta(\mathfrak{m}) = \Delta(m_1) \otimes \Delta(m_2)$ for any $\mathfrak{m} = (m_1, m_2) \in \mathcal{P}_2(m)$. Also, the formula $\Delta(m) = \Delta(\mathfrak{m}) \circ \Delta_{\mathfrak{m}}$ holds true for any $\mathfrak{m} \in \mathcal{P}_2(m)$, so it is not difficult to see that $\Delta(\mathfrak{m})(x_{\mathfrak{m}}) \in \widetilde{C}_m$. Therefore, we can define a function $\alpha^m : Z^{2,m} \to \widetilde{C}_m$ by the formula $\alpha^m(x) = \Delta(\mathfrak{m})(x_{\mathfrak{m}})$.

Consider the set of 2-coboundaries, $B^{2,m} = \mathrm{Im} d^1_m$. If $x = \{x_{\mathfrak{m}}\}_{\mathfrak{m}\in\mathcal{P}_2(m)}$ is an element of $B^{2,m}$, then there exist $c \in C_m$ such that $x_{\mathfrak{m}} = \Delta^{\mathfrak{m}}(c)$, for any positive 2-partition $\mathfrak{m}$. Therefore, the image of x through α^m is an element of $\overline{C}_m$. Moreover, α^m induces a map which we denote abusively also by $\alpha^m : B^{2,m} \to \overline{C}_m$. By all these remarks and constructions, we get a commutative diagram:

$$\begin{array}{ccc} B^{2,m} & \xrightarrow{\alpha^m} & \overline{C}_m \\ \downarrow & & \downarrow \\ Z^{2,m} & \xrightarrow{\alpha^m} & \widetilde{C}_m \end{array} \tag{2.10}$$

where the vertical arrows are the canonical inclusions.

Now let us continue with a rather general lemma.

Lemma 2.5: *Let* C *be a strongly graded* R*-coring. If* $\overline{C}_m = \widetilde{C}_m$*, then* $\operatorname{Ext}^{2,m}_C(R, R) = 0$.

Proof: Let $x = \{x_{\mathfrak{m}}\}_{\mathfrak{m}\in\mathcal{P}_2(m)}$ be a 2-cocycle. Then $\alpha^m(x) \in \widetilde{C}_m = \overline{C}_m$. Now, since we have $\overline{C}_m \subseteq \operatorname{Im}\Delta(m)$, it follows that there exists $c \in C_m$ such that $\Delta(\mathfrak{m})(x_{\mathfrak{m}}) = \Delta(m)(c)$. From the formula $\Delta(m) = \Delta(\mathfrak{m}) \circ \Delta_{\mathfrak{m}}$, using the fact that $\Delta(\mathfrak{m})$ is injective, we deduce that $x_{\mathfrak{m}} = \Delta_{\mathfrak{m}}(c)$, for any positive 2-partition $\mathfrak{m}$. Therefore, x is a 2-coboundary. □

The characterisation for quadratic corings is given in the following result.

Proposition 2.4: *A strongly graded* R*-coring* C *is quadratic if and only if* $\operatorname{Ext}^{2,m}_C(R, R) = 0$*, for all degrees* $m \geq 3$.

Proof: Using the lemma above, the vanishing of $\operatorname{Ext}^{2,m}_C(R, R)$ is a necessary condition. For sufficiency, using Proposition 2.3 and proceeding as in the proof of the dual case, Proposition 2.2, the components of the map $\phi_C : C \to \widetilde{C}$ are all isomorphisms. □

Finally, the anticipated result follows easily.

Corollary 2.1: *Any Koszul* R*-coring is quadratic.*

Proof: Using the results in the previous proposition and also the equivalence between the statements (1) and (7) in Theorem 2.2, one obtains the required result. □

2.3 Graded Linear Duality for Koszul (Co)Rings

The general preliminaries discussed in §1.1.3 can be applied in the case of Koszul (co)rings to provide very useful results, which we detail right now. In the following chapter of this thesis, these particular results will be discussed in the case of finite graded partially ordered sets (posets) to investigate Koszulity.

We will use the notations and assumptions from the referenced preliminary subsection. The following result shows that graded linear duality does keep almost Koszulity.

Proposition 2.5: *Let* (A, C) *be an almost Koszul pair. If* A *and* C *are left locally finite, then the pair* $({}^{*-gr}C, {}^{*-gr}A)$ *is almost Koszul. Similarly, when* A *and* C *are right locally finite, the pair* (C^{*-gr}, A^{*-gr}) *is also almost Koszul.*

Proof: Let $\theta = \theta_{C,A}$ be the connecting isomorphism from the definition of almost Koszul pairs. We will prove that $({}^{*-gr}C, {}^{*-gr}A)$ is an almost Koszul pair with respect to the transpose map of θ which is ${}^*\theta : {}^*A^2 \to {}^*C_2$. First, it is clear that ${}^*\theta$ is an isomorphism of R^{op}-bimodules from its definition, so we have to prove the equality (1.9). Let $\alpha \in {}^*A^2$ and $c \in C_2$. Then we have:

$$\sum \left({}^*\theta(\alpha_{1,1}) * {}^*\theta(\alpha_{2,1})\right)(c) = \sum {}^*\theta(\alpha_{1,1})\left(c_{1,1}{}^*\theta(\alpha_{2,1})(c_{2,1})\right)$$
$$= \sum \alpha_{1,1}\left(\theta(c_{1,1})\alpha_{2,1}(\theta(c_{2,1}))\right) = \sum \alpha(\theta(c_{1,1})\theta(c_{2,1})) = 0.$$

The first and second equalities follow using the definition of the multiplication on ${}^{*-gr}C$ and the transpose map, respectively. Given the equivalence between the relations (1.7) and (1.6), we get the third equality. Finally, taking into account the relation for θ (equation 1.10) in the definition of almost Koszul pairs, we get the nullity of the result.

Of course, the fact that (A^{*-gr}, C^{*-gr}) is almost Koszul can be proved analogously, so we can skip the details. □

Moreover, we can take this result a step forward and prove that actually a Koszul pair corresponds by left duality to a Koszul pair.

Theorem 2.3: *Let (A, C) be a Koszul pair. If A and C are left locally finite, then $({}^{*-gr}C, {}^{*-gr}A)$ is a Koszul pair. Similarly, if A and C are right locally finite, the pair (C^{*-gr}, A^{*-gr}) is Koszul.*

Proof: Assume that (A, C) is a Koszul pair, so that the subcomplex $K^l_\bullet(A, C, m)$ is exact for $m > 1$ (see §1.2.2 for details). We will prove that the corresponding subcomplex, namely $K_l^\bullet({}^{*-gr}C, {}^{*-gr}A, m)$ is also exact, for all $m > 0$ by means of showing that it is isomorphic to the left graded dual of the former. This, in turn, follows using the diagram below:

$$\begin{array}{ccccccccccc}
0 \to & {}^*A^m \otimes {}^*C_0 & \xrightarrow{\partial^0} & {}^*A^{m-1} \otimes {}^*C_1 & \xrightarrow{\partial^1} \cdots \xrightarrow{\partial^{m-2}} & {}^*A^1 \otimes {}^*C_{m-1} & \xrightarrow{\partial^{m-1}} & {}^*A^0 \otimes {}^*C_m & \to 0 \\
 & \sim\downarrow\phi & & \sim\downarrow\phi & & \sim\downarrow\phi & & \sim\downarrow\phi & \\
0 \to & {}^*(A^m \otimes C_0) & \xrightarrow{\delta^0} & {}^*(A^{m-1} \otimes C_1) & \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{m-2}} & {}^*(A^1 \otimes C_{m-1}) & \xrightarrow{\delta^{m-1}} & {}^*(A^0 \otimes C_m) & \to 0
\end{array}$$

This diagram has commutative squares, where we have denoted by ∂^n and δ^n the differential map of $K_l^\bullet({}^{*-gr}C, {}^{*-gr}A, m)$ and the transpose of the restriction of d^l_{m-n+1} to $A^{n-1} \otimes C_{m-n+1}$, respectively.

Let θ be the structural isomorphism of the (almost) Koszul pair (A, C) and choose $\alpha \in {}^*A^n$, $\beta \in {}^*C_{m-n}$, $a \in A^{n-1}$ and $c \in C_{m-n+1}$. Then the action of the differential ∂^n is given by:

$$\partial^n(\alpha \otimes \beta) = \sum \alpha_{1,n-1} \otimes {}^*\theta(\alpha_{2,1}) * \beta.$$

By the definition of the isomorphism ϕ constructed in the preliminary section on Graded Linear Duality (§§1.1.3), the definition of the left convolution product, right

R-linearity of θ and the equivalence between equations (1.6) and (1.7), we compute:

$$\begin{aligned}\Big((\phi \circ \partial^n)(\alpha \otimes \beta)\Big)(a \otimes c) &= \sum \alpha_{1,n-1}\Big(a({}^*\theta(\alpha_{2,1} * \beta)(c)\Big) \\ &= \sum \alpha_{1,n-1}\Big(a\alpha_{2,1}(\theta(c_{1,1}\beta(c_{2,m-n}))\Big) \\ &= \sum \alpha_{1,n-1}\Big(a\alpha_{2,1}(\theta(c_{1,1})\beta(c_{2,m-n}))\Big) \\ &= \sum \alpha\Big(a\theta(c_{1,1})\beta(c_{2,m-n})\Big).\end{aligned}$$

Now, since δ^n is the transpose of the restriction of the differential d^l_{m-n+1} to the space $A^{n-1} \otimes C_{m-n+1}$, the definition of ϕ enables us once again to compute:

$$\begin{aligned}\Big((\delta^n \circ \phi)(\alpha \otimes \beta)\Big)(a \otimes c) &= \phi\Big(\alpha \otimes \beta)(d^l_{m-n+1}(a \otimes c)\Big) \\ &= \phi(\alpha \otimes \beta)\Big(\sum a\theta(c_{1,1}) \otimes c_{2,m-n}\Big) = \alpha\Big(a\theta(c_{1,1})\beta(c_{2,m-n})\Big).\end{aligned}$$

One can compare the results of the two sets of computations above and deduce that the squares of the diagram from the beginning of the proof are commutative, as required. For the result on the right side, as in the previous cases, the argumentation follows similarly and will be skipped.

□

An immediate corollary of this theorem is equally important, since it provides us a similar result, but without the framework of (Koszul) pairs, although the proof will still make use of these tools.

Corollary 2.2: *Let A be a connected and graded R-ring. If A is left (right) locally finite and Koszul, then its left (right) graded linear dual is a Koszul R^{op}-coring.*

Proof: Recall that from the definition, it can be easily seen that the n degree component of the shriek coring of A, namely $A^!_n$ is a submodule of the finitely generated left R-module $A^1 \otimes A^1 \otimes \ldots \otimes A^1$, the tensor product having n factors. Because R is a Noetherian ring, it follows that $A^!$ must be left locally finite. But since A is a Koszul ring, we know that $(A, A^!)$ is a Koszul pair (cf. Theorem 2.1), so using the theorem above, we take the left graded dual and deduce that $({}^{*-gr}(A^!), {}^{*-gr}A)$ is a Koszul pair. In particular, it follows that ${}^{*-gr}A$ is a Koszul R^{op}-coring. □

By duality, one can immediately prove the following:

Corollary 2.3: *Let C be a connected and graded R-coring. If C is left (right) locally finite and Koszul, then its left (right) graded linear dual is a Koszul R^{op}-ring.*

In the classical study of Koszul rings, one studies *Koszul duals*. In the framework that we are using, for a Koszul ring A, one can regard $T(A)$ as its Koszul dual and equally for a Koszul coring C, the coring $E(C)$ can be seen as its Koszul dual. The fact that these structures are legitimate duals can be seen via Corollary 1.2. The following result shows the interplay between these notions of duality and those of graded linear duality, which is particularly well-behaved.

Corollary 2.4: *Let A be a Koszul R-ring and C be a Koszul R-coring.*

(1) If A *and* C *are left locally finite, then* $E({}^{*-gr}A) \simeq {}^{*-gr}T(A)$ *and* $T({}^{*-gr}C) \simeq {}^{*-gr}E(C)$*.*

(2) If A *and* C *are right locally finite, then* $E(A^{*-gr}) \simeq T(A)^{*-gr}$ *and* $T(C^{*-gr}) \simeq E(C)^{*-gr}$*.*

Proof: As previously, we will prove the result on one side, the argument for the other being skipped by reasons of deep similitude. Assume that A is left locally finite. Since A is Koszul, by Theorem 2.1, the pair $(A, T(A))$ is Koszul and also there is a canonical isomorphism $A^! \simeq T(A)$. But as in the proof of the previous corollary, $A^!$ is left locally finite, which makes $T(A)$ left locally finite as well and then $({}^{*-gr}T(A), {}^{*-gr}A)$ is also Koszul. But using [JPS, Theorem 2.9] (cited here as Theorem 1.2), for any Koszul pair (B, D), the graded corings D and $T(B)$ must be isomorphic. In particular, for our pair, we obtain readily that $E({}^{*-gr}A) \simeq {}^{*-gr}T(A)$. □

The main application of these results which we will discuss will be in the field of incidence rings for posets, for which we will study Koszulity and introduce a constructive algorithm for building up on a poset which has a Koszul incidence ring, such that it preserves Koszulity. But we postpone this presentation for the following chapter which we devote to applications.

Now we turn our attention to enriching the equivalent characterisations for Koszul rings and bring into discussion the Ext ring associated to a Koszul ring.

2.4 The Ext Ring of a Koszul Ring

We still keep the assumption of left local finiteness on all R-rings and corings in this section, so that we can work with left graded duals without concern.

For an almost Koszul pair (A, C), we will recall briefly the setup which is needed. Let $K^l_\bullet(A, C) = A \otimes C_\bullet$ be the Koszul complex in the category of left A-modules and $\beta^l_\bullet(A) = A \otimes A^\bullet_+$ be the normalised bar resolution of A. Define a map of complexes $\phi_\bullet : K^l_\bullet(A, C) \to \beta^l_\bullet(A)$ by $\phi_{-1} = \mathrm{Id}_R$, the identity on $A \simeq A \otimes C_0 = K^l_0(A, C) = \beta^l_0(A)$ and for all nonzero degrees $n \geq 1$ and elements $a \in A, c \in C_n$, by the formula:

$$\phi_n(a \otimes c) = \sum a \otimes \theta(c_{1,1}) \otimes \theta(c_{2,1}) \otimes \ldots \otimes \theta(c_{n,1}),$$

which is basically the composition of the iterated comultiplication $\Delta(n)$ on $C_n \to C_1^{(n)}$ with an n-fold tensor product of the isomorphism θ with itself.

Using this definition, it follows that $\phi_\bullet$ is a morphism of complexes, which lifts the identity of R, as explained in §1.2.1.

Now let M be an arbitrary left R-module. By definition, the normalised chain complex is $\Omega^n(A, M) = \mathrm{Hom}_R(A^n_+, M)$ and $K^n_l(A, M) = \mathrm{Hom}_R(C_n, M)$, for all $n \geq 0$. We want to compute the cohomology of these complexes using the Ext functor. Thus, applying the contravariant functor $\mathrm{Hom}_A(-, M)$ to $\phi_\bullet$ and using the adjunction theorem, we get a morphism of complexes:

$$\phi^\bullet : \Omega^\bullet(A, M) \to K^\bullet_l(A, M),$$

whose action on a map $f : A_+^{(n)} \to M$ and an element $c \in C_n$ is given by:

$$\phi^n(f)(c) = f\Big(\sum \theta(c_{1,1}) \otimes \theta(c_{2,1}) \otimes \ldots \otimes \theta(c_{n,1})\Big).$$

We are interested in a particular case, namely that when $M = R$. Note that in this case, the differential ∂^n of the first complex is null. Indeed, the action of A on R is trivial, so for $c \in C_{n+1}$, $f \in \mathrm{Hom}_R(C_n, R)$, we have:

$$\partial^n(f)(c) = \sum \theta(c_{1,1})f(c_{2,n}).$$

In particular, when taking the cohomology, it follows that $\phi^\bullet$ induces a map from ${}^{*-gr}C = \oplus_{n\geq 0}\mathrm{Hom}_R(C_n, R)$ to $\mathrm{Ext}^\bullet_A(R, R) = \mathcal{E}(A)$. This last ring is the newcomer and will be the main object of study in this section, along with its interplay with the other structures connected to Koszul pairs. We will call it simply *the* Ext *ring* of A.

Let us prove now a more general lemma which will be used immediately after in a result that connects $\mathcal{E}(A)$ with ${}^{*-gr}C$.

Lemma 2.6: *(a) Let* C *be a differential graded (DG) coring. Then the left graded dual of it,* ${}^{*-gr}C$ *is a DG ring as well.*

(b) If $f : C \to D$ *is a morphism of DG corings, then its transpose,* ${}^*f : {}^{*-gr}D \to {}^{*-gr}C$ *is a morphism of DG rings.*

Altogether, ${}^{*-gr}(-)$ *is a functor from the category of DG corings to that of DG rings.*

Proof: We already know that the left graded dual of an R-ring is an R^{op}-coring, so we are left with proving that the multiplication of ${}^{*-gr}C$ is compatible with the differential. Hence, let d denote the differential of C, then ${}^*d$, its transpose, is the differential of ${}^{*-gr}C$. We know that d is a coderivation, i.e. if Δ is the comultiplication of C and c is an element of C_{n+m+1}, the following relation holds true:

$$\delta d(c) = \sum d(c_{1,n}) \otimes c_{2,m} + (-1)^n \sum c_{1,n} \otimes d(c_{2,m}).$$

We now have to prove that the transpose of d is a derivation, i.e. it satisfies the Leibniz rule with respect to the graded convolution:

$${}^*d(\alpha * \beta) = ({}^*d\alpha) * \beta + (-1)^n \alpha * {}^*d\beta,$$

for all $\alpha \in \mathrm{Hom}_R(C_n, R)$, $\beta \in \mathrm{Hom}_R(C_m, R)$. Recall that the graded convolution product is defined by:

$$(\alpha * \beta)(c) = \sum \alpha\Big(c_{1,n}\beta(c_{2,n})\Big),$$

for any α, β as above and $c \in C_{n+m}$.

Now we compute the sides of the equality to be proven:

$$(({}^*d\alpha) * \beta)(c) = \sum {}^*d\alpha(c_{1,n+1}\beta(c_{2,m})) = \sum \alpha(d(c_{1,n+1})\beta(c_{2,m})).$$

The other summand is:

$$(\alpha * ({}^*d\beta))(c) = \sum \alpha(c_{1,n}({}^*d\beta)(c_{2,m+1}))) = \sum \alpha(c_{1,n}\beta(d(c_{2,m+1}))).$$

Finally,

$$({}^*d(\alpha * \beta))(c) = (\alpha * \beta)(dc) = \sum \alpha(d(c_{1,n+1})\beta(c_{2,m})) + (-1)^n \sum \alpha(c_{1,n}\beta(d(c_{2,m+1}))).$$

The first equality follows by the definition of ${}^*d$ and the second, by the coderivation property of d and the definition of the convolution product.

The second part of the lemma is easily proved starting from the definitions. We know that the transpose of a graded coring map is a graded ring map, so we only have to show that it is compatible with the differentials. Hence, let d_C and d_D be the corresponding differentials of the DG corings C and D, respectively. Then, since f is a morphism of DG corings, it implies that $f \circ d_C = d_D \circ f$. Using the definitions of the transpose maps, we obtain readily that ${}^*f \circ {}^*d_D = {}^*d_C \circ {}^*f$, as required and the proof is complete. □

The aforementioned result can now be stated and proved.

Proposition 2.6: *Let (A, C) be an almost Koszul pair. There exists a canonical morphism of graded R-rings $\mathcal{E}(A) \to {}^{*-gr}C$, the first being endowed with the cup product, while the second, with the graded convolution product.*

Proof: Recall that there is a morphism of complexes: $\phi_\bullet : K^l_\bullet(A, C) \to \beta^l_\bullet(A, C)$. Delete the component of degree -1 and apply the functor $(-) \otimes_A R$ and we get an induced morphism of DG corings, which we will still denote by $\phi_\bullet : K^l_\bullet(A, R) \to \Omega_\bullet(A)$.

As remarked previously, the first DG coring is simply $C_\bullet$ with the zero differential. By the first part of the lemma above, ${}^{*-gr}C$ is a DG ring. The second part of the lemma, along with the immediate identification ${}^{*-gr}\Omega_\bullet(A) = \Omega^\bullet(A, R)$ complete the proof by passing to cohomology. □

A more powerful result relates $\mathcal{E}(A)$ to $T(A)$, by refining the hypothesis of the proposition as follows:

Theorem 2.4: *If the pair (A, C) is Koszul, then the induced morphism $\phi^\bullet$ is invertible and there is an isomorphism $\mathcal{E}(A) \simeq {}^{*-gr}T(A)$.*

Proof: The supplementary hypothesis, that of (A, C) being a Koszul pair ensures that the map $\phi_\bullet$ defined above is a quasi-isomorphism, being obtained from a morphism between the Koszul and the bar resolutions, which lifts the identity of R. Therefore, when passing to cohomology, the morphism $\phi^\bullet = H_\bullet(\phi_\bullet, R)$ is bijective. Now using [JPS, Theorem 2.9] (cited here as Theorem 1.2), $C \simeq T(A)$ and $A \simeq E(C)$. Since we know that both of the structures are left locally finite (from the underlying assumption of this section), taking left graded duals preserves the isomorphism. Thus, we obtain the required result. □

Corollary 2.5: *The isomorphism in the theorem can be further expanded to $\mathcal{E}(A) \simeq E({}^{*-gr}A)$.*

Proof: If (A, C) is a Koszul pair, then by Corollary 2.3, ${}^{*-gr}T(A) \simeq E({}^{*-gr}A)$. Plugging this isomorphism into the theorem, by transitivity, we obtain readily $\mathcal{E}(A) \simeq E({}^{*-gr}A)$, as needed. □

We can formulate yet another result which uses the Ext ring to characterise Koszulity. In fact, there is a more general result which we can prove.

Lemma 2.7: *Let R be a semisimple ring and V, W be two R-bimodules which are finitely generated as left R-modules. A morphism of R-bimodules $f : V \to W$ is injective if and only if its left dual ${}^*f = \mathrm{Hom}_R(f, R)$ is surjective.*

Proof: Denote by $X = \mathrm{Ker} f$, hence the sequence $0 \to X \to V \xrightarrow{f} W$ is exact. Given the semisimplicity of the ring R, it follows that R is an injective R-module. Therefore, the functor $\mathrm{Hom}_R(-, R)$ is contravariant exact and produces an exact sequence:

$$^*W \xrightarrow{^*f} {}^*V \to {}^*X \to 0.$$

We deduce that $^*X = \mathrm{Coker} f$.

Now we can make an essential remark: $X = 0 \Leftrightarrow {}^*X = 0$. To see this, note first that if $X = 0$, the result holds trivially. Conversely, assuming that $^*X = 0$, we will prove that if $X \neq 0$, then one can construct a nonzero left R-linear map $X \to R$ (i.e. an element of $^*X$), thus obtaining a contradiction.

Assume there exists a nonzero element $x \neq 0$ in X. There is an injective mapping $Rx \to X$ and using the injectivity of the left R-module R, we can extend this map to $X \to R$. Let $\varphi : R \to Rx$ be the canonical map $1 \mapsto x$. Then $\mathrm{Ker}\varphi$ is a left R-submodule of R and there is an isomorphism $R/\mathrm{Ker}\varphi \simeq Rx$.

Note that if $\mathrm{Ker}\varphi = R$, then φ must be the null morphism, but this contradicts the isomorphism of $R/\mathrm{Ker}\varphi$ with Rx, because x was assumed to be nonzero.

To prove that also $\mathrm{Ker}\varphi \neq R$, since R is semisimple, it follows that $\mathrm{Ker}\varphi$ must be a direct summand of R, so there exists a left R-submodule J such that $R \simeq \mathrm{Ker}\varphi \oplus J$. We have thus far the sequence:

$$Rx \simeq R/\mathrm{Ker}\varphi \simeq J \hookrightarrow R,$$

so we can define $\psi : Rx \to R$ as being the composition of the last isomorphism with the inclusion. Using again the injectivity of R as a left R-module, we can complete the diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & Rx & \longrightarrow & X \\ & & \psi \downarrow & \swarrow g & \\ & & R & & \end{array}$$

Thus, the nonzero map $g : X \to R$ provides a contradiction and finishes the proof. □

The lemma above can be used to prove another result regarding the graded linear dual of an R-ring. We refer the reader to the previous section for details on the definitions and requirements regarding this subject. For any nonzero positive integers m, n, consider the diagram:

$$\begin{array}{ccc} ^*A_{n+m} & \xrightarrow{^*\mu_{n,m}} & ^*(A_n \otimes A_m) \\ & \searrow^{\Delta_{n,m}} & \downarrow \\ & & ^*A_n \otimes {}^*A_m \end{array}$$

The vertical arrow is an isomorphism and $^*\mu_{m,n}$ is the transpose of the multiplication of A. Also, as usual, $\Delta_{m,n}$ is the comultiplication of the R^{op}-coring $^{*-gr}A$. Note that if the component A^p is finitely generated as an R-bimodule, for all $p \in \mathbb{N}$, then $A^n \otimes A^m$ must also be finitely generated, for all positive integers m, n.

The following result can be proved in this context using the lemma.

Proposition 2.7: *The R-ring A is strongly graded if and only if the R^{op}-coring $^{*-gr}A$ is so.*

Proof: From the last diagram above, it suffices to note that A is strongly graded if and only if $\mu^{m,n}$ is surjective for all $m, n \in \mathbb{N}$, by definition. This, in turn, holds if and only if ${}^*\mu_{n,m}$ is injective and $\Delta_{n,m}$, which is its composition with an isomorphism, is also injective. Finally, according to the definition, this makes the R^{op}-coring ${}^{*-gr}A$ strongly graded. □

If, furthermore, A is Koszul, we can produce another isomorphism which uses the Ext ring.

Theorem 2.5: *Let* A *be a connected, left locally finite* R*-ring. If* A *is Koszul, then* $\mathcal{E}(A)$ *is also Koszul and there exists an isomorphism* $\mathcal{E}(A) \simeq ({}^{*-gr}A)^!$.

Proof: The result will easily follow putting together some previous statements. As such, using Theorem 2.1, A is a Koszul R-ring if and only if $T(A)$ is a Koszul R-coring. This is further equivalent to ${}^{*-gr}T(A)$ being Koszul, by Corollary 2.3. The isomorphism $\mathcal{E}(A) \simeq {}^{*-gr}T(A)$ from Theorem 2.4 makes $\mathcal{E}(A)$ a Koszul R-ring. Since ${}^{*-gr}A$ is also Koszul, using Theorem 2.2, we know that $E({}^{*-gr}A) \simeq ({}^{*-gr}A)^!$. Now from Corollary 2.5, the result follows. □

We close this section with a remark. If $\mathcal{E}(A)$ is a Koszul R-ring, then, in particular, it is also strongly graded. This means that it is generated by its homogeneous component of degree 1 or, equivalently, that the components of its multiplication (which is the cup product) are all surjective. Under this assumption, $\cup^{p,q} : \mathcal{E}^p(A) \otimes \mathcal{E}^q(A) \to \mathcal{E}^{p+q}(A)$ are surjective, for all $p, q \in \mathbb{N}$.

The interplay between Koszulity and the Ext ring can be further put together in a last simple, yet comprehensive corollary, whose proof follows directly from the remark above and the results formulate thus far.

Corollary 2.6: *If* A *is a Koszul* R*-ring, then* $T(A)$ *is a strongly graded* R*-coring and* $\mathcal{E}(A)$ *is a strongly graded* R*-ring.*

CHAPTER 3

APPLICATIONS

In this chapter, we will present two classes of applications for the theoretical results developed thus far. Both of them will enable us to provide concrete examples, as well as computational illustrations.

We will address the study of partially ordered sets (hereafter called simply *posets*), which are finite and graded, to which we will associate an R-ring and an R-coring, obtained as natural extended constructions for the usual incidence algebra and path coalgebra. The section devoted to the so-called *Koszul posets* (i.e. posets for which the incidence R-(co)ring is Koszul) are filled with concrete examples and illustrations to support the theoretical results that we will state and prove, but also to show the limitations of our presentation by mentioning several exceptions.

In the second part of the chapter, we will discuss *monoid rings* associated to submonoids of $\mathbb{Z}^n$, the n-fold direct sum of the integers. In this case, we will make us of our results regarding the Ext ring for a Koszul ring. Concretely, if A is a monoid ring as above, we will provide a concrete computational example for $\mathcal{E}(A)$ and give a presentation of it with generators and relations.

So let us start by recalling the basic preliminaries regarding posets and incidence (co)rings that can be associated to them. We must add that the general framework is well-known in the literature and that a study in a more general (ungraded) framework was performed in [RS]. Also, a first study for Koszulity of incidence algebras of posets can be found in [Wo].

3.1 Koszul Posets

Apart from the well-known introductions and studies of incidence algebras for posets, we will present here the subject framed in the setting that we have been using so far in this thesis. As such, after the general recollection of concepts related to multiplicative structures associated to finite (and graded) posets, we will see how one can fit everything in the setting of Koszul pairs. The first step will be endowing the incidence algebra of a (finite, graded) poset with an R-ring structure, where the base (semisimple) ring will be a direct sum of a field with itself.

3.1.1 General Framework and Theory

The general framework is as follows. Let $(\mathcal{P}, \leq)$ be a finite poset. A restriction of the binary relation "$\leq$" will also be used and by $x < y$ in $\mathcal{P}$ we will mean that $x \leq y$, but of necessity $x \neq y$. For any two comparable elements $x \leq y$ in $\mathcal{P}$, denote by $[x, y]$ the set $[x, y] = \{z \in \mathcal{P} \mid x \leq z \leq y\}$. If there is no middle element, i.e. $[x, y] = \{x, y\}$, then we say that x is a *predecessor* of y and that y is a *successor* of x. An increasing sequence in $[x, y]$ of the form $x_0 < x_1 < \cdots < x_p$ such that $x_0 = x$ and $x_p = y$ will be called a *chain*. Moreover, the chain is called *maximal* if x_i is a predecessor of x_{i+1} (for all $i = 0, \ldots, p$) such that the chain spans the whole closed interval $[x, y]$ in $\mathcal{P}$. The number p is called the *length* of the chain.

Of special interest here are the so-called *graded posets* , that is those posets for which every interval of its maximal chains has the same length. We will denote by $l([x, y])$ the length of an interval $[x, y]$ and in the graded setting, we will refer to it as *the* length of $[x, y]$, provided the chain is maximal. In general, the set of all intervals of length p will be denoted by $\mathcal{I}_p$.

Now starting from the basic construction of the incidence algebra of a poset, one can turn it into an R-ring as follows. Let $\mathbb{k}$ be a field and $\mathcal{P}$ be a finite poset (the binary relation will always be denoted hereafter with "$\leq$"). Denote the $\mathbb{k}$-linear space with the basis $\mathcal{B} = \{e_{x,y} \mid x \leq y\}$ by $\mathbb{k}^a[\mathcal{P}]$. This vector space becomes an associative and unital $\mathbb{k}$-algebra using the well-known product given by:

$$e_{x,y} \cdot e_{z,u} = \delta_{y,z} e_{x,u},$$

where $\delta_{y,z}$ denotes the Kronecker symbol and the multiplication is extended linearly.

Note that the set $\{e_{x,x}\}_{x \in \mathcal{P}}$ is a set of orthogonal idempotents, $e_{x,x} \cdot e_{x,x} = e_{x,x}$, $\forall x \in \mathcal{P}$ and $e_{x,x} \cdot e_{y,y} = 0, \forall x \neq y$ in $\mathcal{P}$. Therefore, they span a subalgebra R, which is in fact isomorphic to $\mathbb{k}^n$, where n is the cardinality of the poset $\mathcal{P}$. Hence, provided that $\mathcal{P}$ is graded, $\mathbb{k}^a[\mathcal{P}]$ becomes a graded and connected R-ring. Here, the homogeneous component of degree p is the linear span of the elements $e_{x,y}$, for $l([x, y]) = p$. In this setting, $\mathbb{k}^a[\mathcal{P}]$ will be referred to as *the incidence* R*-ring of* $\mathcal{P}$.

By duality, one can construct a coalgebra corresponding to a finite poset. We will denote it by $\mathbb{k}^c[\mathcal{P}]$ and recall that as underlying vector space it coincides with $\mathbb{k}^a[\mathcal{P}]$. The (coassociative and counital) comultiplication is defined on the basis elements by:

$$\Delta(e_{x,y}) = \sum_{z \in [x,y]} e_{x,z} \otimes_{\mathbb{k}} e_{z,y}.$$

The canonical map induced by the inclusion $\mathbb{k} \hookrightarrow R$ gives rise to a canonical morphism $\xi : \mathbb{k}^c[\mathcal{P}] \otimes_{\mathbb{k}} \mathbb{k}^c[\mathcal{P}] \to \mathbb{k}^c[\mathcal{P}] \otimes_R \mathbb{k}^c[\mathcal{P}]$. Then, the composition with the comultiplication, $\xi \circ \Delta : \mathbb{k}^c[\mathcal{P}] \to \mathbb{k}^c[\mathcal{P}] \otimes_R \mathbb{k}^c[\mathcal{P}]$ defines an R-coring structure and thus, hereafter, the unadorned tensor product will be taken over R, as in the rest of the thesis. Moreover, the counit of this coring is given by $\varepsilon : e_{x,y} \mapsto \delta_{x,y} e_{x,x}$.

Therefore, the counit and the comultiplication of this coring will still be denoted by ε and Δ, respectively and the action of the former is simply:

$$\Delta(e_{x,y}) = \sum_{z \in [x,y]} e_{x,z} \otimes e_{z,y}.$$

Given the well-known duality that exists between the incidence coalgebra (coring) and the incidence algebra (ring) of a poset $\mathcal{P}$, the following result which connects their Koszulity should be expected:

Theorem 3.1: *Keeping the notation and assumptions from above, there is an isomorphism* $\Bbbk^a[\mathcal{P}] \simeq {}^{*-gr}\Bbbk^c[\mathcal{P}]$.

In particular, the R*-ring* $\Bbbk^a[\mathcal{P}]$ *is Koszul if and only if the* R*-coring* $\Bbbk^c[\mathcal{P}]$ *is Koszul.*

Proof: Let us denote, for simplicity, $A = \Bbbk^a[\mathcal{P}]$ and $C = \Bbbk^c[\mathcal{P}]$. Therefore, we have to prove that there exists an isomorphism $A \simeq {}^{*-gr}C$. Let $x \leq y$ be two comparable elements and define a $\Bbbk$-linear map $f_{x,y} : C_p \to R$ by $f_{x,y}(e_{u,v}) = \delta_{x,u}\delta_{y,v}e_{u,u}$, for all $u \leq v$ with $l([u,v]) = p$. In fact, $f_{x,y}$ is also left R-linear and if $f \in {}^*C_p$ with $[x,y] \in \mathcal{I}_p$, then there exists a scalar $\alpha_{x,y} \in \Bbbk$ such that $f(e_{x,y}) = \alpha e_{x,x}$, where f is a generalisation of $f_{x,y}$, irrespective of the elements in the subscript. Therefore, it follows that the the components $f_{x,y}$ can be put together into a morphism:

$$f = \sum_{[x,y]\in\mathcal{I}_p} \alpha_{x,y} f_{x,y}.$$

This way one sees that f can be written uniquely as a linear combination of the elements $\{f_{x,y} \mid [x,y] \in \mathcal{I}_p\}$, which makes the former a linear basis of ${}^*C_p$.

Now let $x \leq y, z \leq t$ and $v \leq w$. Then, a direct computation using the graded convolution product shows that:

$$(f_{x,y} * f_{z,t})(e_{v,w}) = \begin{cases} e_{v,v}, & \text{if } y = z, x = v \text{ and } t = w; \\ 0, & \text{otherwise.} \end{cases}$$

But, by the definition, $f_{x,t}(e_{v,w}) = \delta_{x,v}\delta_{t,w}e_{v,v}$ so $f_{x,y} * f_{z,t} = \delta_{y,z}f_{x,t}$, for all x, y, z, v, w as above.

In summary, the $\Bbbk$-linear map $\chi_p : A^p \to {}^*C_p$ given by $\chi_p(e_{x,y}) = f_{x,y}, \forall x \leq y$ such that $l([x,y]) = p$ is the component of degree p for an isomorphism of graded R-rings.

By Corollaries 2.2 and 2.3, if C is a Koszul coring, then $A \simeq {}^{*-gr}C$ is a Koszul ring. Both A and C are locally finite, as finite dimensional vector spaces. Therefore, $C \simeq ({}^{*-gr}C)^{*-gr} \simeq A^{*-gr}$, so C is Koszul provided that A is as well. □

For simplicity, let us give the following definition:

Definition 3.1: A finite graded poset $\mathcal{P}$ will be called *Koszul* if its incidence ring $\Bbbk^a[\mathcal{P}]$ is Koszul (equivalently, $\Bbbk^c[\mathcal{P}]$ is Koszul).

For a graded poset $\mathcal{P}$, let V denote the homogeneous component of degree 1 for its incidence ring. Let $[x,y] \in \mathcal{I}_2$ and define the element:

$$\zeta_{x,y} = \sum_{z\in(x,y)} e_{x,z} \otimes e_{z,y}.$$

Denote by $I_{\mathcal{P}}$ the ideal generated in $T_R^a(V)$ by the set $\{\zeta_{x,y} \mid l([x,y]) = 2\}$. Having established this, the following result will prove useful:

Theorem 3.2: *Let* A *be the incidence ring of a Koszul poset* $\mathcal{P}$*. The* R*-ring* $T_R^A(V)/I_{\mathcal{P}}$ *is Koszul.*

Proof: Using Theorem 3.1, the incidence coring $C = \Bbbk^c[\mathcal{P}]$ is Koszul. Therefore, from the characterisations provided in Theorem 2.2, the R-ring $C^!$ is Koszul as well. But let us note that $C^! = T^a_R(V)/I_{\mathcal{P}}$, as $\zeta_{x,y} = \Delta_{1,1}(e_{x,y})$, whenever $[x, y]$ is an interval of length 2 and this remark completes the proof. □

Finally, we can formulate another result which brings into discussion the use of the Ext ring for a Koszul ring, the latter being associated to a Koszul poset.

Proposition 3.1: *Let $(\mathcal{P}, \leq)$ be a Koszul poset. Denote by $A = \Bbbk^a[\mathcal{P}]$ and by $C = \Bbbk^c[\mathcal{P}]$ the incidence R-ring and the incidence R-coring respectively for this poset. Then there is an isomorphism of graded R-rings:*

$$\mathcal{E}(\Bbbk^a[\mathcal{P}]) \simeq \Bbbk^c[\mathcal{P}]^!,$$

the latter being the shriek ring associated to the incidence R-coring of $\mathcal{P}$.

Proof: We refer to the result of Theorem 2.5 and also Theorem 3.1 above. From there we deduce the isomorphism $\Bbbk^a[\mathcal{P}] \simeq {}^{*-gr}\Bbbk^c[\mathcal{P}]$. Since $\mathcal{P}$ is a Koszul poset, it follows that the pair (A, C) is a Koszul pair. By Theorem 2.1, we know that the pair $(A, A^!)$ is Koszul as well and that there exists a canonical isomorphism $A^! \to T(A)$. Moreover, from the uniqueness (up to isomorphism) of the "match" in a Koszul pair ([JPS, Corollary 2.10 and Theorem 2.13]), we deduce that $T(A) \simeq C$ and the pair $(C^!, C)$ is also Koszul (cf. Theorem 2.2). Putting all of these together, in Theorem 2.5, we obtain the required result. □

The use of the interplay with the Ext ring is that this result brings a simplification for computing $\mathcal{E}(A)$ when A is the incidence ring of a (Koszul) poset. More precisely, it enables us to present this structure in a generators and relations form. This can be performed using the isomorphism with the shriek ring implied in the Theorem above, as follows. Let Γ be any finite quiver associated to the poset $\mathcal{P}$ representing its Hasse diagram. Using the isomorphism in the theorem, we can provide the following explicit description:

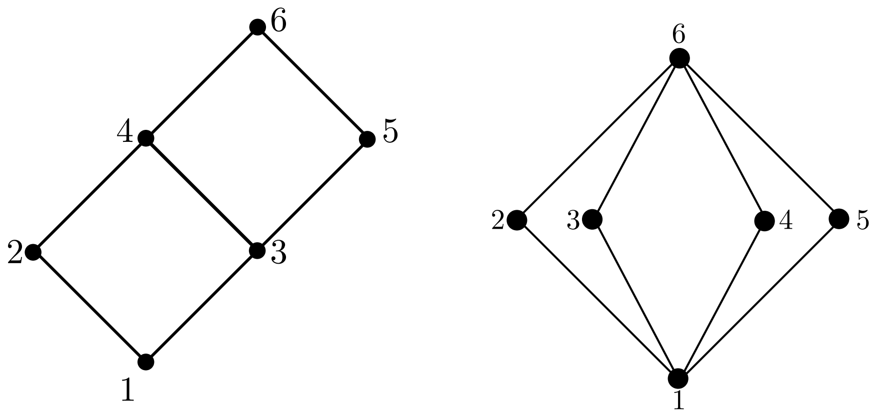
$$\mathcal{E}(\Bbbk^a[\mathcal{P}]) \simeq \Bbbk^c[\mathcal{P}]^! = \frac{\Bbbk^a[\Gamma]}{\left\langle \sum_{x<z<y} e_{x,z} \otimes e_{z,y} \mid l([x,y]) = 2 \right\rangle}. \tag{3.1}$$

For computational reasons, let us anticipate a bit and discuss two classes of examples which we will detail in the very following subsection. As such, consider the two posets depicted in Figure 3.1.

The first poset is Koszul (as it will follow from the very next subsection) and thus the incidence algebra of the corresponding quiver $A = \Bbbk^a[\Gamma]$ is generated by the set $\mathcal{B} = \{e_{1,2}, e_{1,3}, e_{2,4}, e_{3,4}, e_{3,5}, e_{4,6}, e_{5,6}\}$ The relations are spanned as a $\Bbbk$-vector space $\mathcal{R}$ by the parallel paths (i.e. sharing the source and the target), of length two. Namely, $\mathcal{R} = \langle e_{1,2} \otimes e_{2,4} + e_{1,3} \otimes e_{3,4}, e_{3,4} \otimes e_{4,6} + e_{3,5} \otimes e_{5,6} \rangle$. Therefore, by the identification in equation (3.1), we have the isomorphism:

$$\mathcal{E}(\Bbbk^a[\mathcal{P}]) \simeq \Bbbk^c[\mathcal{P}]^! = \Bbbk^a[\Gamma]/\mathcal{R}.$$

In what concerns the second poset (which is a particular example of a *nested vertical diamond* - see the following section), which is also Koszul, one can use a similar reasoning. As such, the generator set would be $\mathcal{B} = \{e_{1,2}, e_{1,3}, e_{1,4}, e_{1,5}, e_{2,6}, e_{3,6}, e_{4,6}, e_{5,6}\}$. In



(a) Planar Koszul tiling (b) Nested Koszul diamond

Figure 3.1: Concrete examples of Koszul posets

this case, the ideal of relations is generated by one element:

$$\mathcal{R} = \Big\langle \sum_{i=2}^{5} e_{1,i} \otimes e_{i,6} \Big\rangle.$$

3.1.2 Concrete Constructions

We will now provide some examples of classes of posets which are Koszul. All of them will be obtained algorithmically by adjoining a greatest element to a given subset of a Koszul poset (in particular, to the trivial singleton poset). The Koszulity of the resulting poset will be ensured by some homological restrictions to a module that is canonically associated to that given subset.

But first, let us prove that a way of obtaining new Koszul rings from old ones is by means of direct sums.

Proposition 3.2: *Let R and S be semisimple rings and A, C be connected R-rings, B, D be connected S-rings.*

(1) The $R \times S$-ring $A \times B$ is Koszul if and only if A and B are Koszul.

(2) Dually, $C \oplus D$ is a Koszul $R \times S$-coring if and only if C and D are Koszul.

Proof: The proof is somehow tedious and technical, but very much straightforward. For an R-bimodule V and an S-bimodule W, the abelian group $V \oplus W$ is a bimodule over $R \times S$, with respect to the componentwise actions:

$$(r, s) \cdot (v, w) = (rv, sw) \quad \text{and} \quad (v, w) \cdot (r', s') = (vr', ws'),$$

for all elements pertaining to the corresponding sets. Moreover, there is an isomorphism of left $A \times B$-modules of the form:

$$(A \times B) \otimes_{R \times S} (V \oplus W) \simeq (A \otimes_R V) \oplus (B \otimes_S W), \tag{3.2}$$

the $A \times B$-action on the direct sum on the right hand side given component-wise. The isomorphism is given by mapping $(a, b) \otimes_{R\times S} (v, w) \mapsto (a \otimes_R v, b \otimes_S w)$.

Now, the comultiplication of the coproduct $A^! \oplus B^!$ is given by:

$$\Delta_{p,q}(x,y) = \sum (x_{1,p}, 0) \otimes_{R\times S} (x_{2,q}, 0) + \sum (0, y_{1,p}) \otimes_{R\times S} (0, y_{2,q}).$$

And also, the pair $(A \times B, A^! \oplus B^!)$ is almost Koszul with respect to the isomorphism $\theta = \theta_{A,A^!} \oplus \theta_{B^!,B}$, as the following computation implies (for any $(x,y) \in A^!_2 \oplus B^!_2$):

$$\left(\mu^{1,1} \circ (\theta \otimes_{R\times S} \theta) \circ \Delta_{1,1}\right)(x,y) = \sum (\theta_{A^!,A}(x_{1,1})\theta_{A^!,A}(x_{2,1}, 0)) + \\ + \sum \left(0, \theta_{B^!,B}(y_{1,1})\theta_{B^!,B}(y_{2,1})\right) = 0.$$

The isomorphism in equation (3.2) enables us to obtain an isomorphism of complexes:

$$K^l_\bullet(A \times B, A^! \oplus B^!) \simeq K^l_\bullet(A, A^!) \oplus K^l_\bullet(B, B^!),$$

which follows easily by a similar computation to that one above.

The conclusion is immediate: $(A \times B, A^! \oplus B^!)$ is Koszul if and only if $(A, A^!)$ and $(B, B^!)$ are so. Therefore, the first part of the Theorem follows using the characterisations in Theorem 2.1.

For the second part, one can proceed dually in a very similar way. □

Let us now fix a field $\Bbbk$ and a finite graded poset $\mathcal{P}$. Assume $\mathcal{P}$ has a maximal element, t and let $\mathcal{Q} = \mathcal{P} - \{t\}$. Denote by $\mathcal{F}$ the set of all predecessors of t in $\mathcal{P}$. The incidence ring of $\mathcal{P}$ will be denoted by A and the corresponding one for $\mathcal{Q}$ will be called B. In fact, A is an R-ring, where $R = \sum_{x\in\mathcal{P}} \Bbbk e_{x,x}$, while B is an S-ring, denoting by $S = \sum_{x\in\mathcal{Q}} \Bbbk e_{x,x}$. As usual, we have denoted the basis elements for A by $\{e_{x,y}\}_{x,y\in\mathcal{P}}$ and the respective ones for B, namely $\{e_{x,y}\}_{x,y\in\mathcal{Q}}$.

Finally, denote by M the linear span in A of all elements $e_{x,t}$ with $x < t$. Therefore, $e_{x,t} \in M \Leftrightarrow x \leq u$ for some $u \in \mathcal{F}$. Now we make an easy remark: M is a B-submodule of A and one can decompose M as:

$$M = \sum_{u\in\mathcal{F}} Be_{u,t}.$$

Now recall that by an n-chain in $\mathcal{P}$ we mean a sequence $\mathbf{x} = (x_0, \dots, x_n)$ of elements in $\mathcal{P}$ such that $x_0 < x_1 < \cdots < x_n$. We will borrow a term from quiver theory and call x_n the *target* of the chain, denoted by $t(\mathbf{x})$. For the length of the chain, by definition, we have $l(\mathbf{x}) = \sum_{i=0}^{n-1} l([x_i, x_{i+1}]) = l([x_0, x_n])$. The set of n-chains of length m from $\mathcal{P}$ will be denoted by $\mathcal{P}_{n,m}$ and we let $\mathcal{P}_n$ denote the whole set of n-chains, regardless of their length. Also in what terminology and notation is concerned, for $\mathbf{x} \in \mathcal{P}_n$, we define an element $e_\mathbf{x} \in A_+^{\otimes_\Bbbk n}$ to be:

$$e_\mathbf{x} = e_{x_0,x_1} \otimes_\Bbbk e_{x_1,x_2} \otimes_\Bbbk \otimes_\Bbbk \dots \otimes_\Bbbk e_{x_{n-1},x_n}.$$

Let us now prove a technical result which will enable us to study Koszulity of the incidence rings discussed so far.

Theorem 3.3: *In the context and notations above, the following isomorphism holds true:*

$$\mathrm{Tor}^A_{n,m}(R,R) \simeq \mathrm{Tor}^B_{n,m}(S,S) \oplus \mathrm{Tor}^B_{n-1,m}(S,M). \tag{3.3}$$

In particular, A *is Koszul if and only if* B *is Koszul and* $\mathrm{Tor}^{B}_{n-1,m}(S,M)=0$*, for all* $n\neq m$ *and any* B*-bimodule* M.

Proof: We start by recalling that the Tor spaces are bigraded, the second index corresponding to the fact that the poset $\mathcal{P}$ is graded. We have seen in §1.1.4 that one can compute $\mathrm{Tor}^{A}_{n,m}$ as the nth homology group for the complex $\Omega_\bullet(A,m)$. Also recall that A_+ is the direct sum of $\Bbbk e_{x,y}$, where x,y are arbitrary elements in $\mathcal{P}$ such that $x<y$.

Now, using the fact that $\Bbbk e_{x,y}\otimes_R \Bbbk e_{x',y'}=0$, whenever $y\neq n'$, it follows that

$$\Omega_n(A,m)=(A_+^{\otimes_R n})_m=\bigoplus \Bbbk e_{x_0,x_1}\otimes_R \Bbbk e_{x_1,x_2}\otimes_R\ldots\otimes_R \Bbbk e_{x_{n-1},x_n},$$

a direct sum of R-bimodules, where $\mathbf{x}=(x_0,\ldots,x_n)$ is a chain in $\mathcal{P}_{n,m}$. Furthermore, for any n-chain of length m, there is an isomorphism of the form:

$$\Bbbk e_{x_0,x_1}\otimes_R \Bbbk e_{x_1,x_2}\otimes_R\ldots\otimes_R \Bbbk e_{x_{n-1},x_n}\simeq \Bbbk e_{x_0,x_1}\otimes_\Bbbk \Bbbk e_{x_1,x_2}\otimes_\Bbbk\ldots\otimes_\Bbbk e_{x_{n-1},x_n}=\Bbbk e_{\mathbf{x}}.$$

Therefore, we can identify the term $\Omega_n(A,m)$ with the $\Bbbk$-linear subspace of the n-th tensor power of A_+, which is actually spanned by all $e_{\mathbf{x}}$, with $\mathbf{x}\in\mathcal{P}_{n,m}$. Using this identification, the differential of the complex $\Omega_\bullet(A,m)$ becomes:

$$d_n(e_{\mathbf{x}})=\sum_{i=1}^{n-1}(-1)^{i-1}e_{x_0,x_1}\otimes_\Bbbk\ldots\otimes_\Bbbk e_{x_{i-1},x_{i+1}}\otimes_\Bbbk\ldots\otimes_\Bbbk e_{x_{n-1},x_n}.$$

Similarly, we can identify the subcomplex $\Omega_n(B,m)$ with the subspace of $A_+^{\otimes_\Bbbk n}$ spanned by the elements $e_{\mathbf{x}}$, with $\mathbf{x}\in\mathcal{Q}_{n,m}$. And this way, $\Omega_\bullet(B,m)$ can be seen as a subcomplex of $\Omega_\bullet(A,m)$.

Let us now denote by Ω'_n the $\Bbbk$-subspace of $A_+^{\otimes_\Bbbk n}$ generated by $e_{\mathbf{x}}$, with $\mathbf{x}\in\mathcal{P}_{n,m}$ and also $t(\mathbf{x})=t$. Keep in mind that $x_0,\ldots,x_{n-1}$ are elements of $\mathcal{Q}$. Moreover, $\Omega'_\bullet$ is a subcomplex of $\Omega_\bullet(A,m)$ from the remarks above.

Now, since t is maximal in $\mathcal{P}$, it follows that $\Omega_\bullet(A,m)=\Omega_\bullet(B,m)\oplus\Omega'_\bullet$ and in particular, when computing the homology of the latter, we get:

$$\mathrm{Tor}^{A}_{n,m}(R,R)=\mathrm{Tor}^{B}_{n,m}(S,S)\oplus H_n(\Omega'_\bullet). \tag{3.4}$$

In order to compute the remaining homology group from the equation above, note that $\Omega'_\bullet\simeq(\beta^r_{\bullet-1}(B)\otimes_B M)_m$, where $\beta^r_\bullet(B)$ is the normalised bar resolution of S in the category of right B-modules. This isomorphism is derived using the successive identifications:

$$\beta^r_{n-1}(B)\otimes_B M\simeq(B_+^{\otimes_S n-1}\otimes_S B)\otimes_B M\simeq B_+^{\otimes_S n-1}\otimes_S M.$$

Therefore, reasoning as above, we can identify the homogeneous component of degree m of $B_+^{\otimes_S n-1}\otimes_S M$ with the subspace of $A_+^{\otimes_\Bbbk n}$ generated by $e_{\mathbf{x}}\otimes_\Bbbk e_{t(\mathbf{x}),t}$, where $\mathbf{x}\in\mathcal{Q}_{n-1}$ and $l(\mathbf{x})+l([t(\mathbf{x}),t])=m$. The former linear space is actually Ω'_n and thus there is an isomorphism $\Omega'_n\simeq(\beta^r_{n-1}(B)\otimes_B M)_m$. Moreover, these isomorphisms are compatible with the differentials, so they define an isomorphism of complexes, in fact. In conclusion, we have a way of computing the homology of $\Omega'_\bullet$:

$$H_n(\Omega'_\bullet)\simeq H_n\Big((\beta^r_{\bullet-1}(B)\otimes_B M)_m\Big)\simeq\mathrm{Tor}^{B}_{n-1,m}(S,M).$$

Therefore, this completes the proof of the isomorphism in equation (3.3) and the last claim concerning the Koszulity of A is immediate using Theorem 2.1. □

The following result is also used in constructing the applications:

Lemma 3.1: *In the context and notations above,* $\mathrm{Tor}_n^A(R, Ae_{u,u}) = 0$ *for all* $u \in \mathcal{P}$ *and* $n > 0$. *Furthermore,* $\mathrm{Tor}_{n,m}^A(R, Ae_{u,u}) = 0$, *for all* $n \geq 0$ *and* $n \neq m$.

Proof: Let $n > 0$. Since $e_{u,u}$ is an idempotent in A, it follows that $Ae_{u,u}$ is a projective A-module, so there is a decomposition $A = Ae_{u,u} \oplus A(1 - e_{u,u})$. Since the component $\mathrm{Tor}_{n,m}^A(R, Ae_{u,u})$ is a direct summand in $\mathrm{Tor}_n^A(R, Ae_{u,u})$, which is zero given the projectivity of $Ae_{u,u}$, this implies that the former is also zero, for all m.

We only have left to prove that $\mathrm{Tor}_{0,m}^A(R, Ae_{u,u}) = 0$, for all $m > 0$. By the definition, $\mathrm{Tor}_{0,m}^A(R, Ae_{u,u}) = (R \otimes_A Ae_{u,u})_m$ and using the following isomorphisms, the result is readily obtained:

$$(R \otimes_A Ae_{u,u})_m \simeq \left(\frac{Ae_{u,u}}{A_+(Ae_{u,u})}\right)_m \simeq \left(\frac{Ae_{u,u}}{A_+e_{u,u}}\right)_m \simeq (Re_{u,u})_m = 0.$$

□

Remark 3.1: Note that the nullity of $\mathrm{Tor}_{n,m}^A(R, Ae_{u,u})$ holds true for any poset $\mathcal{P}$ and any element $u \in \mathcal{P}$, provided that $m \neq n$. In particular, keeping the notations that we have used so far, we also have that $\mathrm{Tor}_{n-1,m}^B(S, Be_{s,s}) = 0$, for any $s \in \mathcal{Q}$ and $m \neq n$.

Yet another useful result is the following.

Lemma 3.2: *In the context and notations above,* $\mathrm{Tor}_{n-1,m}^B(S, Be_{u,t}) = 0$, *for any element* $u \in \mathcal{F}$ *and* $m \neq n$.

Proof: We start by remarking that the sets $\{e_{x,u} \mid x \leq u\}$ and $\{e_{x,t} \mid x \leq t\}$ are bases for the vector spaces $Be_{u,u}$ and $Be_{u,t}$ respectively. Therefore, the unique correspondence $e_{x,u} \mapsto e_{x,t}, \forall x \leq u$ is linear and bijective. On the other hand, this map is also B-linear, so with respect to internal gradings, since $\deg(e_{x,t}) = \deg(e_{x,u}) + 1$, the map induces a graded isomorphism of degree +1 between $Be_{u,u}$ and $Be_{u,t}$. It follows that there is an isomorphism at the Tor level:

$$\mathrm{Tor}_{n-1,m}^B(S, Be_{u,t}) \simeq \mathrm{Tor}_{n-1,m-1}^B(S, Be_{u,u}).$$

The proof finishes by using the previous lemma and the remark that followed it. □

As we have mentioned throughout the thesis, we will provide a constructive algorithm which will enable obtaining a Koszul poset starting with one that has the same property (in particular, the singleton $\mathcal{P} = \{\bullet\}$ could be a starting point). In order to apply the algorithm which we will describe, one has to check some simple combinatorial conditions on the poset. Let us now describe these conditions and their base and method of application.

Definition 3.2: Using the notations and definitions that were valid thus far, we say that the set $\mathcal{F}$ *satisfies the condition* (†) if either $\mathcal{F}$ is a singleton or there is a common predecessor s of all elements in $\mathcal{F}$ such that $s = \inf\{u, v\}, \forall u \neq v \in \mathcal{F}$. Equivalently, the condition happens if and only if:

$$Be_{u,t} \cap Be_{v,t} = Be_{s,t}. \tag{3.5}$$

The condition may seem cumbersome and somehow unnatural at this point, but the use of it will become evident throughout the proof of the next result. It is also worth noting that this condition (†) is purely combinatorial, depending only on the structure of the poset $\mathcal{F}$ and its subset $\mathcal{F}$. This aspect makes it easy to check whether a starting poset obeys it.

Theorem 3.4: *Let $\mathcal{P}$ be a finite graded poset and $t \in \mathcal{P}$ a maximal element. Let A and B denote the incidence algebras for $\mathcal{P}$ and $\mathcal{Q} = \mathcal{P} - \{t\}$ respectively. If $\mathcal{F}$ is the set of all predecessors of t in $\mathcal{P}$ and $\mathcal{F}$ obeys the condition (†), then A is a Koszul R-ring if and only if B is a Koszul S-ring.*

Proof: In view of Theorem 3.3, it is enough to check that $\mathrm{Tor}^B_{n-1,m}(S, M) = 0$, for all $n \neq m$. We will proceed by induction on f, the cardinality of $\mathcal{F}$.

For $f = 1, M = Be_{u,t}$, where $\{u\} = \mathcal{F}$. The conclusion follows using Lemma 3.2, without the use of the condition (†), which in fact would be automatically true, since the singleton is a special case of it.

Assume that the result is true for any set $\mathcal{F}'$ of cardinality f and let $\mathcal{F}$ be a set with $f + 1$ elements, obeying the condition (†). Take $u \in \mathcal{F}$ and $\mathcal{F}; = \mathcal{F} - \{u\}$. It is clear that $\mathcal{F}'$ satisfies the condition (†), with respect to the same common predecessor s of its elements. Therefore, if we let $M' = \sum_{v\in\mathcal{F}'} Be_{v,t}$, then $\mathrm{Tor}^B_{n-1,m}(S, M') = 0$, for all $n \neq m$.

Denote by $\overline{M} = M/M'$. Then the exact sequence:

$$\cdots \to \mathrm{Tor}^B_{n-1,m}(S, M') \to \mathrm{Tor}^B_{n-1,m}(S, M) \to \mathrm{Tor}^B_{n-1,m}(S, \overline{M}) \to \cdots$$

tells us that it suffices to prove the nullity of the last term, wherefrom the conclusion would follow. In order to do this, note that we can make the following successive identifications:

$$\overline{M} \simeq \frac{Be_{u,t}}{Be_{u,t} \cap \sum_{x\in\mathcal{F}'} Be_{x,t}} \simeq \frac{Be_{u,t}}{Be_{s,t}}.$$

The last identification follows by an immediate generalisation of relation (3.5) from the definition of the (†) condition. Indeed, taking an element $w \in Be_{u,t} \cap \sum_{x\in\mathcal{F}'} Be_{x,t}$, then we can also write $w = \sum_{y\leq u} \alpha_y e_{y,t} = \sum_{x\in\mathcal{F}'} \sum_{z\leq x} \beta_{z,x} e_{z,t}$. For every y such that the coefficient $\alpha_y \neq 0$, the element $e_{y,t}$ must also appear in the double sum with a nonzero coefficient. In particular, there must exist some $x \in \mathcal{F}'$ such that $y \leq x$. But since $y \leq u$ as well, by transitivity we get $e_{y,t} \in Be_{u,t} \cap Be_{x,t} = Be_{s,t}$. Therefore, $w \in Be_{s,t}$ and as a conclusion, $Be_{u,t} \cap \sum_{x\in\mathcal{F}'} Be_{x,t}$ is a submodule of $Be_{s,t}$. The other inclusion is obvious, so the intersection above and $Be_{s,t}$ coincide, as claimed.

Now consider the following exact sequence:

$$\cdots \to \mathrm{Tor}^B_{n-1,m}(S, Be_{u,t}) \to \mathrm{Tor}^B_{n-1,m}\left(S, \frac{Be_{u,t}}{Be_{s,t}}\right) \to \mathrm{Tor}^B_{n-2,m}(S, Be_{s,t}) \to \cdots.$$

We know by Lemma 3.2 that the first term is null and to conclude the proof we have to show that the last one is zero as well.

Proceeding as in the proof of Lemma 3.2, we can provide an isomorphism of of graded B-modules, of degree +2 $Be_{s,s} \to Be_{s,t}$, given by $e_{x,s} \mapsto e_{x,t}, \forall x \leq s$. Therefore:

$$\mathrm{Tor}^B_{n-2,m}(S, Be_{s,t}) \simeq \mathrm{Tor}^B_{n-2,m-2}(S, Be_{s,s}).$$

Hence, by Lemma 3.1, we get the nullity of $\mathrm{Tor}^{B}_{n-2,m}(S, Be_{s,t})$ and this finishes the proof. □

For a thorough description and application of the algorithm which will follow, it is useful to consider the dual case as well. Note that the dual poset of $\mathcal{P}$ coincides, as a set, with $\mathcal{P}$, but $x \leq y$ in $\mathcal{P}$ if and only if $x \geq y$ in the dual. Therefore, we state the dual of condition $(\dagger)$ taking into consideration this reverse relation.

Definition 3.3: A subset $\mathcal{F} \subseteq \mathcal{P}$ satisfies the condition $(\ddagger)$ if and only if either $\mathcal{F}$ is a singleton or there is a common successor s of all elements in $\mathcal{F}$ such that $s = \sup\{u, v\}$, for all $u \neq v$ in $\mathcal{F}$.

We know that an R-ring is Koszul if and only if its opposite R^{op}-ring is Koszul, so working with the dual poset of $\mathcal{P}$, one immediately obtains the theorem below:

Theorem 3.5: *Let $\mathcal{P}$ be a finite graded poset and let $t \in \mathcal{P}$ be a minimal element. Let A and B denote the incidence algebras of $\mathcal{P}$ and $\mathcal{Q} = \mathcal{P} - \{t\}$, respectively. If the set of all successors of t in $\mathcal{P}$ obeys condition $(\ddagger)$, then A is a Koszul R-ring if and only if B is a Koszul S-ring.*

Now let us describe the algorithm, whose validity is ensured by Theorems 3.4 and 3.5 above.

The algorithm starts with any Koszul poset $\mathcal{Q}$ and one could apply any of the constructions below:

(1) Take $u \in \mathcal{Q}$ and adjoin a new element t to $\mathcal{Q}$. On the union $\mathcal{P} = \mathcal{Q} \cup \{t\}$ take the partial ordering defined such that t is a successor of u, but incomparable with any other element of $\mathcal{Q}$.

(2) Take $u \in \mathcal{Q}$ and adjoin a new element t to $\mathcal{Q}$. On the union poset $\mathcal{P} = \mathcal{Q} \cup \{t\}$ define the partial ordering such that t is a common predecessor of u, but incomparable with all the other elements in $\mathcal{Q}$.

(3) Choose a subset $\mathcal{F} \subseteq \mathcal{Q}$ that obeys the condition $(\dagger)$ and adjoin a new element t to $\mathcal{Q}$. On the union poset $\mathcal{P} = \mathcal{Q} \cup \{t\}$, take the partial ordering defined such that t is a successor of all elements in $\mathcal{F}$ and incomparable with all the others of $\mathcal{Q}$.

(4) Choose a subset $\mathcal{F} \subseteq \mathcal{Q}$ which satisfies the condition $(\ddagger)$ and adjoin a new element t to $\mathcal{Q}$. Define the union poset $\mathcal{P} = \mathcal{Q} \cup \{t\}$ whose partial ordering is taken such that t is a predecessor of all elements in $\mathcal{F}$ and incomparable with all the other elements of $\mathcal{Q}$

The algorithm implies that repeating any of the constructions (1)–(4) above finitely many times, after each iteration replacing $\mathcal{Q}$ by $\mathcal{P}$ and the output poset will be Koszul.

In order to make things clearer and more visual, since we mentioned that both the conditions $(\dagger) - (\ddagger)$ and the algorithm are combinatorial in nature and easy to apply, let us detail a bit the steps outlined. The first step allows one to enlarge the poset by adding a "branch" above, while the second adds a "root" below, both connected solely to a chosen element $u \in \mathcal{Q}$. The third construction of the algorithm allows one to "put rooftops" over any subset $\mathcal{F}$ which is depicted in the Hasse diagram on a horizontal. The fact that $\mathcal{F}$ satisfies condition $(\dagger)$ means that all of its elements are "rooted" on the same infimum. The last alternative of the algorithm comes as a dual of the third, in the sense that it allows to "root" all the elements in a horizontal subset $\mathcal{F}$ of $\mathcal{Q}$, which obeys condition (‡) in the sense that all of its elements are "covered" by a same supremum.

The pictorial representation in Figure 3.2 below should further clarify and make more intuitive these constructions.

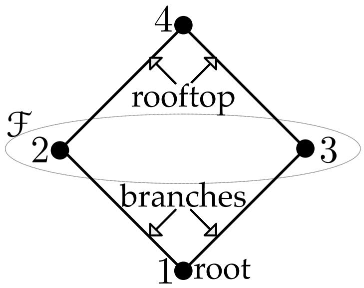


Figure 3.2: Adding branches and rooftops via the algorithm

Given the steps outlined in the algorithm, we distinguish two main types of Koszul posets, which we discuss in detail: *planar tilings* and *(nested) diamonds*.

The first concrete case that we discuss is of posets whose Hasse diagrams look like **planar tilings**, with square tiles for simplicity and symmetry. Let us call a *tile* any poset of the form $\mathcal{T} = \{s_{\mathcal{T}}, u_{\mathcal{T}}, v_{\mathcal{T}}, t_{\mathcal{T}}\}$ such that the elements are all points in the complex plane given by coordinates $s_{\mathcal{T}} = (p, q-1), u_{\mathcal{T}} = (p-1, q), v_{\mathcal{T}} = (p+1, q)$ and respectively $t_{\mathcal{T}} = (p, q+1)$, for some integers p, q. When represented geometrically, they form a square arranged as a kite, that is with the diagonal being parallel to the vertical axis. Therefore, the order relation is defined such that $s_{\mathcal{T}} < u_{\mathcal{T}} < t_{\mathcal{T}}$ and $s_{\mathcal{T}} < v_{\mathcal{T}} < t_{\mathcal{T}}$ but $u_{\mathcal{T}}$ and $v_{\mathcal{T}}$ are incomparable.

As a starting point, one can prove immediately that the poset $\mathcal{T}$, constructed above, is Koszul. For this, start with the most basic poset, let $\mathcal{Q} = \{s_{\mathcal{T}}\}$ which is clearly Koszul. Then the points $u_{\mathcal{T}}$ and $v_{\mathcal{T}}$, as well as the edges that connect them with $s_{\mathcal{T}}$ are added, using the first step in the algorithm. At this stage, we have the poset $\mathcal{P} = \{s_{\mathcal{T}}, u_{\mathcal{T}}, v_{\mathcal{T}}\}$ which looks like a "V" and which is Koszul. In this poset, it is clear that $s_{\mathcal{T}} = \inf\{u_{\mathcal{T}}, v_{\mathcal{T}}\}$ and $u_{\mathcal{T}}$ and $v_{\mathcal{T}}$ are incomparable, so taking $\mathcal{F} = \{u_{\mathcal{T}}, v_{\mathcal{T}}\}$, a subset that satisfies condition (†), adjoin $t_{\mathcal{T}}$ to $\mathcal{P}$ and conclude that the resulting poset, $\mathcal{T}$ is Koszul.

Terminology-wise, we will call any finite union of tiles a *planar tiling* . Of course, we will say that x is a predecessor of y if this is the case in any of the tiles that make up the overall planar tiling.

Let us continue with a warning: not all planar tilings are Koszul. As such, consider the poset depicted in Figure 3.3 below.

To see this, take $\{e_{a,b} \mid a \leq b\}$ as the canonical basis for the coring $C = \Bbbk^c[\mathcal{P}]$. Then:

$$\Delta_{1,1}(e_{s,y}) \otimes e_{y,t} = e_{s,v} \otimes e_{v,y} \otimes e_{y,t} = e_{s,v} \otimes \Delta_{1,1} \otimes e_{v,t}.$$

Therefore, it follows that $\xi = e_{s,v} \otimes e_{v,y} \otimes e_{y,t}$ is an element of the set $\widetilde{C_3}$, which can be written as $(C_1 \otimes \mathrm{Im}(\Delta_{1,1})) \cap (\mathrm{Im}(\Delta_{1,1}) \otimes C_1)$. But on the other hand, $\xi \notin \overline{C_2}$, which is $\mathrm{Im}(\Delta(3))$ and therefore the coring is not quadratic, much less Koszul (see §2.1 for more details).

In fact, there is also a simple direct proof which shows that the incidence R-ring

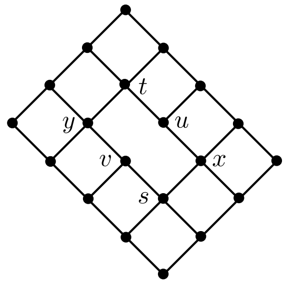


Figure 3.3: Planar tiling which is not quadratic

A of this poset is not Koszul. Although the argumentation is redundant (because of Theorem 3.1), we include it for completeness. Let $\omega = e_{s,y} \otimes e_{y,t} - e_{s,x} \otimes e_{x,t}$ be a 2-cycle in $\Omega_\bullet(A, 3)$. Assume ω is also a 2-coboundary. Since it has the internal degree 3, it can be written as:

$$\omega = d_3\Big(\sum_{(z,z')\in\mathcal{S}} \alpha_{z,z'} e_{s,z} \otimes e_{z,z'} \otimes e_{z',t}\Big),$$

where $\mathcal{S} \subseteq \mathcal{P} \times \mathcal{P}$ is the set of pairs (z, z') such that $[s, z]$, $[z, z']$ and $[z', t]$ are intervals of length 1, while $\alpha_{z,z'}$ is a scalar from $\Bbbk$. Clearly in this example $\mathcal{S} = \{(v, y), (x, u)\}$, hence:

$$\omega = \alpha_{v,y}(e_{s,v} \otimes e_{v,t} - e_{s,y} \otimes e_{y,t}) + \alpha_{x,u}(e_{s,x} \otimes e_{x,t} - e_{s,u} \otimes e_{u,t}).$$

This relations obviously leads to a contradiction, since the four tensor monomials in the right hand side are linearly independent over $\Bbbk$.

Now, in order to obtain Koszul planar tilings, we claim for an inductive argument that the union of a tile $\mathcal{T}$ and a Koszul planar tiling is also Koszul, provided we impose specific restrictions on the intersection $\mathcal{Q} \cap \mathcal{T}$. It is enough to study four separate cases which ensure the Koszulity of $\mathcal{P} = \mathcal{Q} \cup \mathcal{T}$.

(a) If $\mathcal{Q} \cap \mathcal{T} = \emptyset$, then $\Bbbk^a[\mathcal{P}] = \Bbbk^a[\mathcal{Q}] \times \Bbbk^a[\mathcal{T}]$ and using Proposition 3.2, we get that $\Bbbk^a[\mathcal{P}]$ is Koszul.

(b) If $\mathcal{Q} \cap \mathcal{T} = \{u_\mathcal{T}\}$, using the algorithm, one can consider adding successively the elements of $\mathcal{T}$ so at every step we have a Koszul poset. In detail, using construction (1) of the algorithm, one adds $t_\mathcal{T}$ to $\mathcal{Q}$, then $v_\mathcal{T}$ using construction (3). Now, take $\mathcal{F} = \{u_\mathcal{T}, v\}$ and by construction (4), adjoin $s_\mathcal{T}$, since the condition (†) is trivially satisfied for this $\mathcal{F}$. The situation is treated similarly if the intersection is $\{u_\mathcal{T}\}$.

If $\mathcal{Q} \cap \mathcal{T} = \{t_\mathcal{T}\}$ or $\{s_\mathcal{T}\}$, one proceeds by the same idea, i.e. adjoining the components of $\mathcal{T}$ step by step.

(c) When $\mathcal{Q} \cap \mathcal{T} = \{x, y \mid x \leq y \text{ in } \mathcal{T}\}$, then such a set uniquely determines an edge of the Hasse diagram for $\mathcal{T}$. Therefore, the fact that the union $\mathcal{P} = \mathcal{Q} \cup \mathcal{T}$ is Koszul can be proved using the same method, regardless of the choice of $\{x, y\}$. So it is worthy of discussion only in the case when $x = u_\mathcal{T}$ and $y = t_\mathcal{T}$. Thence, using construction (3), one adjoins the element $v_\mathcal{T}$ to $\mathcal{Q}$, then repeat the step outlined in the preceding case to add $s_\mathcal{T}$.

(d) For the case when $\mathcal{Q} \cap \mathcal{T} = \{u_\mathcal{T}, v_\mathcal{T}, x\}$, where either $x = t_\mathcal{T}$ or $x = s_\mathcal{T}$, we proceed as follows. If $x = t_\mathcal{T}$, then by taking $\mathcal{F} = \{u_\mathcal{T}, v_\mathcal{T}\}$ and using construction (4), one can add $s_\mathcal{T}$. But we are left to proving that the set $\mathcal{F}$ satisfies condition (‡). Let us drop the subscript and denote the elements of $\mathcal{T}$ by u, v, t. Consider the maximal sequences of tiles $\{\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n\}$ and $\{\mathcal{T}'_1, \mathcal{T}'_2, \ldots, \mathcal{T}'_m\}$ as depicted in Figure 3.4 below.

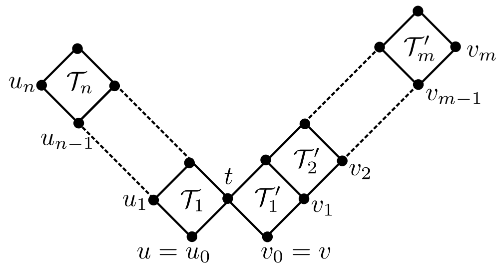


Figure 3.4: Adding tiles

Assume that x is an element in $\mathcal{Q}$ that is greater than or equal to v, then either $x \geq t$ or there exists some $i \in \{0, \ldots, m\}$ such that $x = v_i$. All the elements $x \geq u$ can be characterised in a similar way. Therefore, any upper bound x of $\{u, v\}$ must also be greater than or equal to t, so $t = \sup\{u, v\}$. In particular, when taking a set $\mathcal{F}$ as above, it satisfies the condition (‡). The dual case can be treated analogously.

Let us now provide a concrete compound example. Consider the figure 3.5 below.

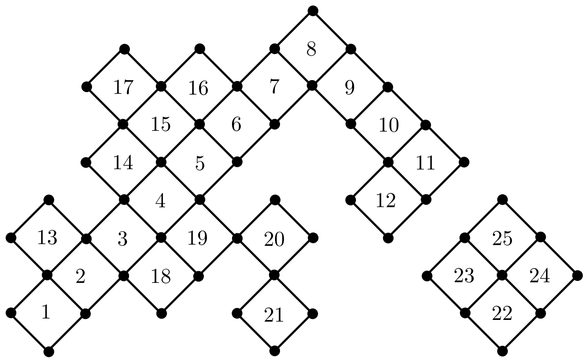


Figure 3.5: A compound Koszul planar tiling

We claim that this compound poset is Koszul and this can be seen by reconstructing it step by step. First off, patch tiles 1 through 8 using the case (c) outlined above. Similarly, tiles 9 to 19 can be added. In order to adjoin tiles 20 and 21, one follows the case (b), since the intersection with the previous construction is made of a sole element. On the other hand, tile 22 is added by case (a), since it has an empty intersection with the tiles numbered 1 through 21. Then 23 and 24 are added by case (c) again, while tile 25 gets added as outlined in case (d).

The basic conclusion which we draw from these cases is that planar tilings which can be constructed using the steps in our algorithm and along the way one encounters only intersections as described in cases (a)-(d) above end up being Koszul. We do not claim that *all* planar tilings are Koszul, since the particular poset discussed previously is not even quadratic. Note however that nor can it be obtained using our algorithm. Furthermore, we also do not claim that *all* Koszul posets are of the form of planar tilings. As we will see immediately, there are at least some two cases which provide, namely **(nested) diamonds**.

Another type of particular Koszul poset which we discuss is that of *nested vertical diamonds*, whose Hasse diagram is depicted in Figure 3.6. Such an example is also a byproduct of the algorithm which we introduced, as follows. Start with the trivial poset $\{s\}$ and adjoin the elements u_i using the first construction of the algorithm. Then by construction, $s = \inf\{u_i, u_j\}_{i\neq j}$ and all such pairs (u_i, u_j) are incomparable. The set $\mathcal{F} = \{u_1, u_2, \ldots, u_n\}$ satisfies condition (†) by construction and one can apply successively construction (3) to add a common "rooftop", that is an element t such that $t \geq u_i, \forall i = 1, \ldots, n$.

As in the case of planar tilings, there is a separate but connected case which we must discuss, this time for the better. Consider a poset $\mathcal{P}_{i,j}$ which could be called a *nested horizontal diamond*, depicted in Figure 3.6 below.

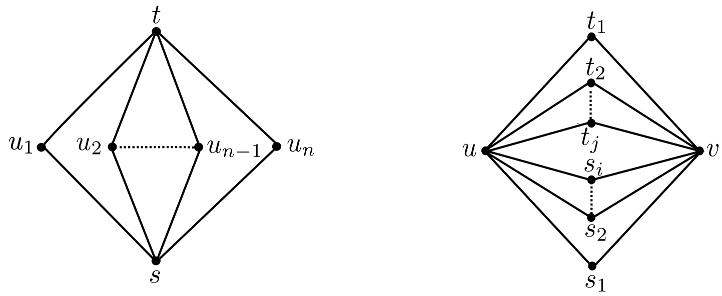


Figure 3.6: Nested vertical and horizontal diamonds, respectively

If $i, j > 1$, this poset cannot be obtained using the algorithm, since neither the infimum, nor the supremum of $\mathcal{F} = \{u, v\}$ exists, so the conditions (†) and (‡) are not valid. However, in the case when $\Bbbk$ is a field of characteristic zero, this poset is also Koszul, but the argument is much different from what we have discussed so far. More precisely, we will prove that $\Bbbk^a[\mathcal{P}_{i,j}]$ is a braided symmetric R-bialgebra. (See §1.4 for the basic definitions and results.)

Let V be the R-bimodule generated by elements $e_{x,y}$ of the canonical basis of $\Bbbk^a[\mathcal{P}_{i,j}]$ such that $[x, y]$ is an interval of length one. Therefore, the set

$$\mathcal{B} = \{e_{s_p,u} \otimes_R e_{u,t_q}, e_{s_p,v} \otimes_R e_{v,t_q} \mid p \leq i \text{ and } q \leq j\}$$

is a basis of the linear space $V \otimes_R V$. Moreover, since there are no maximal chains of length greater than 2, $V^{\otimes_R n} = 0$, whenever $n \geq 3$. Define an R-bilinear braid map $\mathfrak{c} : V \otimes_R V \to V \otimes_R V$ which interchanges the chains of length two that correspond to the same p and q. Concretely, for all $p \leq i < q$, we have:

$$\mathfrak{c}(e_{s_p,u} \otimes_R e_{u,t_q}) = e_{s_p,v} \otimes_R e_{v,t_q} \qquad \text{and} \qquad \mathfrak{c}(e_{s_p,v} \otimes_R e_{v,t_q}) = e_{s_p,u} \otimes_R e_{u,t_q}.$$

The braid equation is trivially satisfied, because $V^{\otimes_R 3} = 0$ and we must also add that $\mathfrak{c}^2 = \mathrm{Id}_{V\otimes_R V}$.

Let I be the ideal generated by $\mathrm{Im}(\mathrm{Id}_{V\otimes_R V} - \mathfrak{c})$. By definition, the symmetric algebra associated to V and $\mathfrak{c}$ is $S_R(V, \mathfrak{c}) = T^a_R(V)/I$. Now we remark that the R-ring $\Bbbk^a[\mathcal{P}_{i,j}]$ coincides with the quotient of $T^a_R(V)$ modulo the ideal generated by differences of the elements in the basis of $V \otimes_R V$, where $p \leq i$ and $q \leq j$. Therefore, $\Bbbk^a[\mathcal{P}_{i,j}] \simeq S_R(V, \mathfrak{c})$ and since the latter is always Koszul (by [JPS, Theorem 6.2]), we get that $\Bbbk^a[\mathcal{P}_{i,j}]$ is actually a Koszul *braided R-bialgebra*.

Remark 3.2: Starting from this slight digression, we can provide another class of Koszul rings that one can associate to nested vertical diamonds. Let $\mathcal{P}$ be such a poset and denote the homogeneous component of degree one of its incidence R-ring by V. Then, using Theorem 3.2, the following R-ring is Koszul:

$$A = T^a_R(V)/R\zeta_{s,t},$$

where $\zeta_{s,t} = \sum_{i=1}^{n} e_{s,u_i} \otimes e_{u_i,t}$ is the element associated to the unique interval $[s,t]$ of length 2 in $\mathcal{P}$.

Remark 3.3: From a terminology standpoint, a clarification is due. In the article [Wo, Definition 4.6], the author defines a graded poset Ω as being *exactly thin* whenever $x < y$ and $l(x,y) = 2$ imply that the interval (x,y) consists of exactly two elements. We remark that what we termed 'planar tilings' are examples of such posets, which we proved to be Koszul. Moreover, the 'nested horizontal diamonds' which we introduced are also exactly thin and Koszul.

Remark 3.4: Let us add that another class of examples can be provided, which are upper triangular matrix coalgebras. Consider the Example 2.9, page 13 of [Iov]. Let $\mathcal{P}$ be the poset $\mathcal{P} = \{x, x_1, \ldots, x_n\}$ with relations $x_i < x$, for all $i = 1, \ldots, n$, while x_i and x_j are any two incomparable. Its incidence R-coring is $C = C_0 \oplus C_1$, where $C_0 = R = \oplus_{y \in \mathcal{P}} \Bbbk e_{y,y}$ and $C_1 = \oplus_{y \in \mathcal{P}} \Bbbk e_{y,x}$. This can be put in the upper triangular matrix form as:

$$C = \begin{pmatrix} R & C_1 \\ 0 & R \end{pmatrix}.$$

Its Hasse diagram is simple:

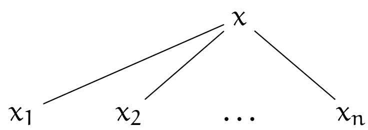


Figure 3.7: A Koszul Upper Triangular Matrix Coalgebra

We remark that this is indeed a Koszul R-coring, since the corresponding poset can be obtained using our algorithm, as follows. Start with $\Omega = \{x_1, \ldots, x_n\}$, all of the elements being any two incomparable, then adjoin x to Ω and define the ordering relation such that $x > x_i$, for all $i = 1, \ldots, n$. Take $\mathcal{F} = \{x_1, \ldots, x_n\}$, which satisfies condition $(\dagger)$ and hence the conclusion follows.

In fact, as it is obvious, this poset can be seen as the upper (or equivalently, lower) half of nested vertical diamonds.

3.2 Integer Monoid Rings

We now discuss another type of examples for Koszul rings, taken from the class of monoid rings in the special case of submonoids of $\mathbb{Z}^n$, the n-fold product of the integer set. This is what we call *integer monoid rings* . In part, we will follow the approach of [RS, §1.2], except for the deduction of the concrete result. More precisely, we will make use of the Ext ring associated to a (Koszul) ring and give a presentation with generators and relations, which we compute explicitly in the case of integer monoid rings.

The basic setup is the following. Let $(M,+)$ be an associative and cancellative monoid. By definition, this is equivalent to the fact that whenever $a + b = a + c$ for some arbitrary elements $a, b, c \in M$, then A can be cancelled and infer that $b = c$. This is *left cancellation* and analogously, we require than whenever $b + a = c + a$, then $b = c$ for *right cancellation*. Let $\Bbbk$ be a field. The *monoid ring* $\Bbbk M$ associated to M is a well known structure, which as a vector space has a basis given by the elements $\{\xi_m\}_{m \in M}$, indexed on M and the multiplication is defined as:

$$\xi_m \cdot \xi_n = \xi_{m+n},$$

which is then extended $\Bbbk$-bilinearly.

We study Koszulity of $A = \Bbbk M$, which could be decomposed as $A = \Bbbk\xi_0 \oplus A_+$, where $A_+ = \Bbbk\xi_m, m \neq 0$. Thus, A becomes an augmented $\Bbbk$-algebra.

Let A^1 be the $\Bbbk$-span of all *indecomposable elements* of M. That is, all those $m \in M$ which cannot be written as $m = m_1 + m_2$ with $m_1, m_2 \in M$, both being nonzero. As mentioned in [RS], it is known that A is generated as a $\Bbbk$-algebra by A^1 and it becomes $\mathbb{N}$-graded if and only if there is a functional on $\mathbb{R}^n$ which takes the value 1 on all indecomposable elements $m \in A^1$. If that should be the case, the homogeneous component of degree n is generated by all elements ξ_m such that m can be written as a sum of exactly n indecomposables in A^1. Let us put $\deg(m) = n$ should this hold.

Let us describe the Ext algebra of a monoid ring $A = \Bbbk M$, in the case when A has a finite number of indecomposable elements. Thus, $A^n = \langle \xi_m \mid \deg(m) = n \rangle$, which makes A finitely generated. Therefore, we can consider the graded (right) linear dual $A^{*-\mathrm{gr}}$, which has a dual basis over the field $\Bbbk$ in the usual sense. Namely, we have $A^*_n = \langle \xi^*_m \mid \deg(m) = n \rangle$. The comultiplication on the graded linear dual is:

$$\Delta : A^{*-\mathrm{gr}} \to A^{*-\mathrm{gr}} \otimes A^{*-\mathrm{gr}}, \quad \Delta(\xi^*_m) = \sum_m \xi^*_{m'} \otimes \xi^*_{m''}.$$

In the definition, we have used implicitly a Sweedler-type notation, in the sense that $m' + m'' = m$, both of the summands being arbitrary.

Now, using the isomorphism in Theorem 2.5, we can provide a presentation of $\mathcal{E}(A)$ with generators and relations. More precisely, $\mathcal{E}(A)$ is the quotient of the tensor algebra of ${}^*A^1$ modulo the ideal generated by $\mathrm{Im}(\Delta_{1,1} : {}^*A^2 \to {}^*A^1 \otimes {}^*A^1)$ (the notation for the dual on the left is simply for readability purposes). Note that when m is an element of degree 2, we have $\Delta(1,1)(\xi^*_m) = \sum \xi^*_{m'} \otimes \xi^*_{m''}$, where m' and m'' are indecomposable and add to m. Therefore, we obtain:

$$\mathcal{E}(A) = \frac{\Bbbk[\xi^*_m \mid \deg(m) = 1]}{\left\langle \Delta_{1,1}(\xi^*_m) \mid \deg(m) = 2 \right\rangle}. \tag{3.6}$$

Koszulity of monoid rings (and more generally, for semigroup rings) is studied in relation to posets in [RS]. We will restrict here to illustrate our results with a concrete case which we discuss in a slightly different manner.

Let M be the submonoid of $\mathbb{Z}^2$ generated by the elements:

$$m_1 = (2,0), m_2 = (0,2), m_3 = (1,1).$$

They all are indecomposable in M, so if we denote by $A = \Bbbk M$, then $A^0 = \Bbbk\xi_0 \simeq \Bbbk$ and A^1 is linearly generated over $\Bbbk$ by $\{m_1, m_2, m_3\}$. It is immediate to see that in this particular case, A is an $\mathbb{N}$-graded $\Bbbk$-algebra.

Moreover, using the equation above we can provide a presentation of A with generators and relations. Since A is generated by A^1 and it has a unique relation, namely $\xi_{m_1} \cdot \xi_{m_2} = \xi_{m_3}^2$, we can identify A with a quotient of a polynomial algebra. That is, $A \simeq \Bbbk[X, Y, Z]/(X^2 - YZ)$. The fact that this is a Koszul algebra can be seen by different methods, for example using complete intersection and the theory of Cohen-Macaulay rings. However, we choose to refer the reader to [PP, Corollary 6.3].

Let us now compute explicitly the Ext algebra of A. Using the presentation in equation (3.6) and making a shorthand notation $\xi_{m_i}^* = x_i$, one obtains readily:

$$\mathcal{E}(A) = \frac{\Bbbk\langle x_1, x_2, x_3\rangle}{x_1^2, x_2^2, x_3^2 + x_1x_2 + x_2x_1, x_1x_3 + x_3x_1}.$$

For a justification of the computations for the relations, recall that it is generated by $\mathrm{Im}\Delta_{1,1}$, whose domain is ${}^*A^2 = \langle \xi_{m_1+m_2}^*, \xi_{m_1+m_3}^*, \xi_{m_2+m_3}^*, \xi_{2m_1}^*, \xi_{2m_2}^*, \xi_{2m_3}^*\rangle$. Now take all the possible decompositions of such elements in ${}^*A^1 \otimes {}^*A^1$ and obtain the respective relations.

3.3 Hochschild (Co)Homology for Koszul Rings

In this section, we will present a basic introduction to computing the Hochschild (co)homology for rings which have the Koszul property. This, along with the section on twisted tensor products, summarized in Theorem 1.4 in relation to Koszulity, are taken from [JPS] and will be put together in an original context in the last section. Therefore, the present section can be seen as an introduction of some tools which will be of use later on.

Let us fix a separable algebra R, over a field $\Bbbk$. The separability ensures the fact that R is a projective R-bimodule and that every R-bimodule is both projective and injective. For more details on separable algebras, see, for example, [We, §9.2]. In this context, for any Koszul pair (A, C), the complex $K_\bullet(A, C)$ is a projective resolution of A in the category of right A-modules, cf. [JPS, Corollary 2.5]. Using this resolution, we will describe the Hochschild (co)homology of A with coefficients in some A-bimodule M and see how this fits in the Koszul framework.

All unadorned tensor products will be taken over R, as before.

Recall that for any algebra R, one can define its *enveloping algebra* $R^e = R \otimes_\Bbbk R^{op}$. Using this, one can see immediately that any R-bimodule can be thought of as an R^e-module, either on the left or on the right. This enveloping algebra is generally used in the computation of the Hochschild (co)homology of an algebra via derived functors, which are to be applied in a module category (see [We, §9.1] for a detailed introduction). Given the fact that R is a $\Bbbk$-algebra, the definition of an R-bimodule V involves an extra condition, that is $xv = vx$, for all $x \in \Bbbk$ and $v \in V$.

Definition 3.4: Let V be an R-bimodule and $[R, V]$ denote the linear span of all commutators $[r, v] = rv - vr$, with $r \in R$ and $v \in V$. Denote by $V_R = V/[R, V]$.

Let $V_1, \ldots, V_n$ be R-bimodules. Their tensor product is an R-bimodule and we define *the cyclic tensor product* of $V_1, \ldots, V_n$ by:

$$V_1 \widehat{\otimes} \ldots \widehat{\otimes} V_n = (V_1 \otimes \ldots \otimes V_n)_R.$$

For the equivalence class of a tensor monomial $v_1 \otimes \ldots \otimes v_n$ in the cyclic tensor product we will use the notation $v_1 \widehat{\otimes} \ldots \widehat{\otimes} v_n$.

Note that for any two R-bimodules V and W, there is an isomorphism between $V\widehat{\otimes}V$ and $V \otimes_{R^e} W$, given by $v\widehat{\otimes}w \mapsto v \otimes_{R^e} w$, therefore, by induction, for all $1 \leq i \leq n$:

$$V_1\widehat{\otimes} \dots \widehat{\otimes} V_n \simeq (V_1 \otimes \dots \otimes V_i) \otimes_{R^e} (V_{i+1} \otimes \dots \otimes V_n).$$

Furthermore, the cyclic tensor product is invariant to cyclic permutations (hence the name). To see this, note that $V\widehat{\otimes}W \simeq W\widehat{\otimes}V$ via the linear "flip" map $v\widehat{\otimes}\, w \mapsto w\widehat{\otimes}v$ and inductively, one obtains that:

$$V_1\widehat{\otimes}\, V_2\widehat{\otimes} \dots \widehat{\otimes}V_n \simeq V_2\widehat{\otimes}V_3\widehat{\otimes} \dots \widehat{\otimes}V_n\widehat{\otimes}V_1 \simeq \dots \simeq V_1\widehat{\otimes}V_1\widehat{\otimes} \dots \widehat{\otimes}V_{n-1}.$$

Using this new object, we can form the corresponding standard complex to compute $HH^\bullet(A, M)$, the Hochschild cohomology of A with coefficients in M. Let A now be a Koszul R-ring, where R is a separable $\Bbbk$-algebra as before. There is a canonical ring morphism from $\Bbbk$ to A, induced by the R-ring structure of A and the fact that R is a $\Bbbk$-algebra. Assume that the image of this map is in the centre of A, which makes A also a $\Bbbk$-algebra. In this context, an A-bimodule can be though of as a left module over the enveloping algebra of A, which is $A^e = A \otimes_{\Bbbk} A^{op}$. Fix a connected R-coring C such that there is a Koszul pair (A, C). Recall that C exists, by Theorem 2.1 and in particular one could take $C = T(A)$. In fact, any such C is isomorphic to $T(A)$ and since the latter is an A^e-module, we can assume that C is as well.

By definition, one computes the Hochschild homology of A with coefficients in M by $HH_\bullet(A, M) = \mathrm{Tor}_\bullet^{A^e}(A, M)$, i.e. by taking a projective resolution of A in the category of A^e-modules and applying the functor $(-) \otimes_{A^e} M$ to the deleted complex, then taking homology.

Let $K'_\bullet(A, C)$ denote the deleted complex of $K_\bullet(A, C)$, that is obtained by dropping the component of degree -1. Therefore, the Hochschild homology of A with coefficients in M becomes the homology of the complex $K'_\bullet(A, C) \otimes_{A^e} M$. We can give a further simplification of this complex, noting that there is an isomorphism:

$$K'_n(A, C) \otimes_{A^e} M \simeq M\widehat{\otimes}C_n \simeq C_n\widehat{\otimes}M,$$

which is given by the map:

$$\varphi_n((x \otimes c \otimes y) \otimes_{A^e} \ \otimes m) = (ymx)\widehat{\otimes}c,$$

for all $x, y \in A, c \in C, m \in M$. Its inverse applies:

$$\varphi_n^{-1}(m\widehat{\otimes}c) = (1 \otimes c \otimes 1) \otimes_{A^e} m.$$

In this new setting, one can define the differential $\partial_n : M\widehat{\otimes}\, C_n \to M\widehat{\otimes}C_{n-1}$ of this complex by the equation:

$$\partial_n = \varphi_{n-1}^{-1} \circ (d_n \otimes_{A^e} \mathrm{Id}_M) \circ \varphi_n. \tag{3.7}$$

We have denoted by d_n the differential of the complex $K_\bullet(A, C)$. More explicitly, the equation above becomes:

$$\partial_n(m\widehat{\otimes}c) = \sum m\theta_{C,A}(c_{1,1})\widehat{\otimes}c_{2,n-1} + (-1)^n \sum \theta_{C,A}(c_{2,1})m\widehat{\otimes}c_{1,n-1}. \tag{3.8}$$

It can be seen that $\varphi_\bullet$ induces an isomorphism of complexes from $(M\widehat{\otimes}C_\bullet, \partial_\bullet)$ to $M \otimes_{A^e} K'_\bullet(A, C)$ and thus we have the following result:

Theorem 3.6: ([JPS, Theorem 3.3]) Let (A, C) be a Koszul pair over a separable $\Bbbk$-algebra R. The Hochschild homology of A with coefficients in an A-bimodule M is the homology of the chain complex $K_\bullet(A, M) = M\widehat{\otimes}C_\bullet$ and the differential is defined by equation (3.8).

As for cohomology, recall that by definition $HH^\bullet(A, M) = \mathrm{Ext}^\bullet_{A^e}(A, M)$ and one can proceed similarly as above to get:

Theorem 3.7: ([JPS, Theorem 3.4]) *Let (A, C) be a Koszul pair over a separable $\Bbbk$-algebra. The Hochschild cohomology of A with coefficients in an A-bimodule M is the cohomology of the cochain complex $K^\bullet(A, M) = \mathrm{Hom}_{R^e}(C_\bullet, M)$. The differential is defined for all $c \in C_{n+1}$ and $f \in \mathrm{Hom}_{R^e}(C_n, M)$ by the equation:*

$$\partial^n(f)(c) = \sum \theta_{C,A}(c_{1,1})f(c_{2,n}) + (-1)^{n+1} \sum f(c_{1,n})\theta_{C,A}(c_{2,1}). \tag{3.9}$$

An interesting application of the results above is the following:

Theorem 3.8: [JPS, Theorem 3.5] *Let (A, C) be a Koszul pair over the $\Bbbk$-algebra R, which we assume to be separable. Then:*

$$\mathrm{Hdim}(A) = \mathrm{pdim}({}_A R) = \mathrm{pdim}(R_A) = \sup\{n \mid C_n \neq 0\},$$

where Hdim denotes the Hochschild dimension and pdim is the projective dimension.

3.4 Twisted Tensor Products

We are now interested in the algebra structure that the Hochschild cohomology has. In general, there is a cup product on $HH^\bullet(A, M)$ which makes it into a $\Bbbk$-algebra, provided that A is an algebra (for any A-bimodule M). We will study this situation for twisted tensor products and moreover, ones which are involved in Koszul pairs. As such, using Theorem 1.4, we know that if (A, C) and (B, D) are Koszul pairs, then $(A \otimes_\sigma B, C \otimes_\tau D)$ is a Koszul pair as well, for some twisting map σ and entwining map τ, invertible and satisfying certain properties.

In this last section, we will study a more particular case, that is starting with a twisting map of R-rings $\sigma : B \otimes A \to A \otimes B$, we will construct an almost-Koszul pair $(A \otimes_\sigma B, T(A) \otimes B)$ in the category of B-bimodules. Then, we will see how the Koszul complexes for the two pairs, $(A, T(A))$ and $(A \otimes_\sigma B, T(A) \otimes B)$ compare, as well as what can be obtained for the Hochschild cohomology rings of the two.

The context is the usual, take R to be a semisimple ring and A a connected, strongly graded R-ring. Let B be an arbitrary R-ring, not necessarily graded, such that the twisting map σ is compatible with the grading on A, that is $\sigma(B \otimes A^p) \subseteq A^p \otimes B$, for all p. Therefore, $A \otimes_\sigma B$ can be seen as a connected B-ring, with respect to the canonical embedding of B into $A \otimes_\sigma B$ given by $b \mapsto 1 \otimes_\sigma b$, for all $b \in B$.

Let C denote the graded connected R-coring $T(A)$, thence (A, C) is an almost Koszul pair. For simplicity, denote by θ the structural isomorphism of the pair. Let $\varphi^{A,B}_\bullet$ be the restriction of the twisting map σ to $B \otimes \overline{A}$. By [JPS, Proposition 4.10], there exists an entwining map $\widetilde{\varphi}^{A,B} : B \otimes \Omega_\bullet(A) \to \Omega_\bullet(A) \otimes B$, which is an isomorphism of chain complexes. Thus, $\widetilde{\varphi}_{\bullet}^{A,B}$ induces an entwining map $\tau : B \otimes C \to C \otimes B$, which respects the grading on C. Moreover, $C \otimes B$ is a B-bimodule with the actions:

$$b \cdot (c \otimes b') = \sum c_\tau \otimes b_\tau b' \quad \text{and} \quad (c \otimes b') \cdot b = c \otimes b'b.$$

Therefore, it is now immediate the fact that the maps:

$$\begin{aligned} &\Delta_C \otimes \mathrm{Id}_B : C \otimes B \to C \otimes C \otimes B \simeq (C \otimes B) \otimes_B (C \otimes B) \text{ and} \\ &\varepsilon_C \otimes \mathrm{Id}_B : C \otimes B \to R \otimes B \simeq B \end{aligned}$$

give $C \otimes B$ a structure of a connected B-coring, both being morphisms of B-bimodules.

As such, the following result is clear:

Theorem 3.9: *The pair* $(A \otimes_\sigma B, C \otimes B)$ *constructed above is almost Koszul and:*

$$K^l_\bullet(A \otimes_\sigma B, C \otimes B) \simeq K^l_\bullet(A, C) \otimes B.$$

Proof: Let $\widetilde{\theta} = \theta \otimes \mathrm{Id}_B$. Given the fact that (A, C) is almost Koszul, it follows that $\widetilde{\theta}$ is an isomorphism between $C_2 \otimes B$ and $A^2 \otimes B$. Let $\widetilde{\Delta} = \Delta_C \otimes \mathrm{Id}_B$ be the comultiplication that makes $C \otimes B$ into a connected B-coring (see above). Then, for $c \in C_2$ and $b \in B$, we have:

$$\begin{aligned} \sum \widetilde{\theta}\Big((c \otimes b)_{1,1}\Big) \cdot_\sigma \widetilde{\theta}\Big((c \otimes b)_{2,1}\Big) &= \sum \widetilde{\theta}(c_{1,1} \otimes 1) \cdot_\sigma \widetilde{\theta}(c_{2,1} \otimes b) \\ &= \sum \theta(c_{1,1}) \otimes \theta(c_{2,1}) \otimes b = 0. \end{aligned}$$

Therefore, the pair $(A \otimes_\sigma B, C \otimes B)$ is Koszul, as claimed. The proof finishes after remarking that the canonical isomorphism:

$$(A \otimes C_\bullet) \otimes B \simeq (A \otimes_\sigma B) \otimes_B (C_\bullet \otimes B)$$

is an isomorphism of complexes, providing the required isomorphism in the theorem. □

Now, in order to address Hochschild cohomology, we assume that B is a separable R-ring. A direct consequence of Theorem 3.7, having in view Theorem 3.9 above as well is the following:

Proposition 3.3: *Keeping the context and the notations above, the Hochschild cohomology group* $HH^\bullet(A \otimes_\sigma B, M)$ *is computed by the complex* $K^\bullet(A \otimes_\sigma B, M) = \mathrm{Hom}_{B^e}(C_\bullet \otimes B, M)$.

Let us describe the differentials of this complex, since the terms are clear. We will use the guidelines of the proof for [JPS, Theorem 3.4], cited above. Take a morphism $f \in \mathrm{Hom}_{B^e}(C_\bullet \otimes B, M)$ and $c \in C_n, b \in B$. Let M be an arbitrary $A \otimes_\sigma B$-bimodule, hence an A-bimodule and a B-bimodule as well, via the natural embeddings of the factors in the twisted tensor product.

Recall that the canonical isomorphism for the Koszul pair $(A \otimes_\sigma B, C \otimes B)$ is simply $\widetilde{\theta} = \theta \otimes \mathrm{Id}_B$. To define the differential of the Hochschild complex, we need a map:

$$\partial^n : \mathrm{Hom}_{B^e}(C_n \otimes B, M) \to \mathrm{Hom}_{B^e}(C_{n+1} \otimes B, M).$$

For this, consider the diagram:

$$\begin{array}{ccccccc}
C_1 \otimes C_n \otimes B & \xrightarrow{\theta \otimes \mathrm{Id}_{C_n} \otimes \mathrm{Id}_B} & A^1 \otimes C_n \otimes B & & \xrightarrow{\mathrm{Id}_{A^1} \otimes f} & & A^1 \otimes M \\
\uparrow {\scriptstyle \Delta^C_{1,n} \otimes \mathrm{Id}_B} & & & & & & \downarrow \\
C_{n+1} \otimes B & & & & & & M \\
\downarrow {\scriptstyle \Delta^C_{n,1} \otimes \mathrm{Id}_B} & & & & & & \uparrow \\
C_n \otimes C_1 \otimes B & \xrightarrow{\mathrm{Id}_{C_n} \otimes \tilde{\theta}} & C_n \otimes A^1 \otimes B & \xrightarrow{\mathrm{Id}_{C_n} \otimes \sigma^{-1}} & C_n \otimes B \otimes_\sigma A^1 & \xrightarrow{f \otimes \mathrm{Id}_{A^1}} & M \otimes A^1
\end{array}$$

The rightmost vertical maps are the module structures of M. Therefore, we can define the differential as being the alternate sum of the two "options" above. More specifically, this acts as:

$$\partial^n(f)(c \otimes b) = \sum \theta(c_{1,1}) f(c_{2,n} \otimes b) + (-1)^{n+1} \sum f(c_{1,n} \otimes b_\sigma)\theta(c_{2,1})_\sigma. \tag{3.10}$$

CHAPTER 4

SUMMARY AND FURTHER RESEARCH

The main scope of research in this thesis was to dualise the results that are known for Koszul R-rings, using the tools provided by *Koszul pairs*, developed in [JPS]. In this sense, the first chapter of the thesis recalls the preliminary notions and results that are used throughout.

In order to proceed with the dualisation procedure, we first provided a definition for the dual structures, namely *Koszul* R*-corings*, then adapted the main results that characterised them. For this purpose, after formulating and proving a comprehensive characterisation theorem (Theorem 2.1), its dual followed, as per Theorem 2.2. Therefore, these results set the theoretical background for further research, which was oriented towards finding as many examples of Koszul R-(co)rings as possible. The second chapter of the thesis presents, thus, the most important characterisation theorems for Koszul rings and corings. Then, in order to apply these notions to some examples, we studied *graded linear duals* for locally finite R-(co)rings. This tool allowed us to prove that a locally finite R-ring A is Koszul if and only if its left (right) graded linear dual ${}^{*-\mathrm{gr}}A$ is a Koszul R-coring (see Theorem 2.3). Then, another theoretical tool that will be used in providing more concrete examples of Koszul (co)rings was introduced, namely the Ext ring associated to a Koszul ring A, denoted by $\mathcal{E}(A)$. This structure is modelled after the Yoneda ring that is associated to an R-ring, but in the Koszul case, it allows us to obtain two isomorphisms which show the interplay of the Ext ring in the theory. More precisely, Theorem 2.4 shows the isomorphism between $\mathcal{E}(A)$ and ${}^{*-\mathrm{gr}}T(A)$. Then, Theorem 2.5 shows that $\mathcal{E}(A)$ is also isomorphic to the *shriek* ring associated to ${}^{*-\mathrm{gr}}A$ (see Definition 1.3 for the definition and the construction of the shriek structure).

The last part of the thesis focuses on applications. Thus, using the theoretical tools developed in the previous chapter(s), we investigate some classes of R-(co)rings which can be seen to be Koszul. A first important class of examples is in the category of finite graded posets, whose incidence algebras (over a field $\Bbbk$) can be made into an R-ring (and also an R-coring, either via a direct construction or via graded linear duality). In this case, in order to provide examples for what we called *Koszul posets* (i.e. posets whose incidence R-(co)ring is Koszul), we provided an algorithm that can be applied after checking some simple combinatorial conditions. For this purpose, subsection 3.1.2 provides the details. We have distinguished two types of posets that can be used, namely *planar tilings* and *nested diamonds*. Moreover, in our theoretical description, we also discuss extensibility and also the limits of our description. In this sense, there exist posets (of the diamond type) that we proved to be Koszul, but which cannot be obtained using our algorithm. Furthermore, not all planar tilings are Koszul posets, as we have showed by providing a counterexample (see Figure 3.3).

Another class of examples that we studied is that of integer monoid rings. That is, starting with submonoids of $\mathbb{Z}^n$ and constructing the associated monoid ring, we can make such structures into R-rings. Using the characterisations via the Yoneda ring associated to an R-ring, in a particular case of a submonoid of $\mathbb{Z}^2$, whose monoid ring we denote by A, we provided a concrete description, by generators and relations, for $\mathcal{E}(A)$, as a quotient of a polynomial algebra.

Lastly, working over a separable $\Bbbk$-algebra R as a base ring, we can investigate the Hochschild cohomology ring for a Koszul R-ring A. Using the cyclic tensor product, we provided an alternate (isomorphic) description of the Hochschild complex for A (cf. Theorem 3.7 and Theorem 3.6) and then went to investigating the ring structure of $HH^\bullet(A, M)$ when A is a twisted tensor product. This is an ongoing research, but at the time being, we have obtained some results in this sense, that are presented in §3.4.

Therefore, throughout this thesis, after a basic recollection of the preliminaries developed mainly in [JPS], we provided original results that not only dualise some results regarding Koszul rings, but they also provide a framework which can be investigated further. Moreover, after the formulation of the essential definitions, we proved a very useful characterisation theorem which shows 7 equivalent descriptions of Koszul (co)rings. At the same time, we investigated examples of classes of Koszul (co)rings, which put into practice the theoretical properties introduced by us and also show some important connections with other branches of mathematics (via, e.g. the theory of linear graded posets).

The results in this thesis are original (unless otherwise specified and cited), and were obtained in the articles [M], [MS1], [MS2].

As subjects for further research, the theory of Koszul (co)rings can be used in many directions. One (that is ongoing) is the study of the Hochschild cohomology ring for a twisted tensor product algebra which forms a Koszul pair. Thus, starting with a Koszul pair (A, C) over a separable algebra R, and taking an arbitrary R-ring B such that there is a twisting map of R-rings $\sigma : B \otimes A \to A \otimes B$, we want to study the Koszulity of the R-ring $A \otimes_\sigma B$, using the first pair (A, C). We already know that $(A \otimes_\sigma B, C \otimes B)$ is an almost Koszul pair and also have some characterisations of the Hochschild complex for $A \otimes_\sigma B$ via the respective one for A.

Another subject of ongoing research is to provide more examples of Koszul (co)rings and for this purpose, as hinted in Remark 3.4, we study (upper) triangular (co)algebras.

Moreover, drawing inspiration from [BG], we are also interested in finding results that connect our approach via Koszul pairs to *deformation theory* for algebras and more particularly, to Koszul (co)algebras and the Poincaré-Birkhoff-Witt theorem.

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